QS 015/2

Matriculation Programme Examination

Semester I

Session 2017/2018

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1. Express $\frac{3x^2-5}{(x-3)(x^2+2)}$ in partial fractions.

2. Solve the equation $\cos\theta + \cos 5\theta = 2\cos 3\theta \ \ for\ 0 \leq \theta \leq \pi$. Give your answers in terms of π.

3. Evaluate the following limits:

   a. $\lim\limits_{x\to 2} \frac{x^3-8}{x^2-2x}$

   b. $\lim\limits_{x\to\infty} \sqrt{\frac{5x+7}{6x-5}}$

4. Given $y = e^{-2x}\sin 3x$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

   Hence, show that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$.

5. Given the polynomial $P(x) = x^2 - 4$ and $Q(x) = \alpha x^4 + x^3 + 2x^2 + \beta x + 28$.

   a. Find all zeroes of $P(x)$.

   b. When $Q(x)$ is divided by $P(x)$, the remainder is $14x + 52$. Use the remainder theorem to find the values of α and β.

   c. Using the values of α and β obtained from part $5(b)$, find the remainder when $2Q(x) + x$ is divided by $P(x)$.

6. Express $\cos\theta + \sqrt{2}\sin\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and α is an acute angle.

   Hence,

   a. Solve the equation $\cos\theta + \sqrt{2}\sin\theta = \frac{\sqrt{3}}{2}$ by giving all solutions between $0°$ and $360°$.

   b. Show the greatest value of $\frac{1}{\cos\theta+\sqrt{2}\sin\theta+5}$ is $\frac{5+\sqrt{3}}{22}$.

7. State the conditions for continuity of $f(x)$ at $x = a$.

   a. By using the conditions for continuity of $f(x)$ at $x = a$, find the values of m and n such that

$$f(x) = \begin{cases} n - 2\cos x, & x < 0 \\ 2 + mx^2, & 0 \leq x < 2 \\ m - x, & x \geq 2 \end{cases}$$

   is continuous on the interval $(-\infty, \infty)$.

b. If $m = -2$ and $n = 4$, determine whether $f(x)$ is differentiable at $x = 2$ or not.

8. A curve with equation $x^2 - 3y^2 = ae^{y-2x} + by - 6$, where a and b are constants, passes through the point $(1, 2)$.

a. Given $\frac{dy}{dx} = 1$ at $(1, 2)$, determine the values a and b.

b. Evaluate $\frac{d^2y}{dx^2}$ at $(1, 2)$.

9. The function f is defined by $f(x) = \frac{\ln(x-1)}{x-1}$ for $x > 1$.

a. By considering the first and second derivatives of $f(x)$, show that there is only one maximum point on the graph $y = f(x)$.

b. Use the result obtained in part 9(a) to state the exact coordinates of the maximum point.

c. Find the x-coordinate of the function f when $\frac{d^2y}{dx^2} = 0$.

10. A curve is defined by the parametric equations $x = 3t - \frac{1}{t}$ and $y = t + \frac{3}{t}$, where $t \neq 0$.

a. Show that $\frac{dy}{dx} = \frac{t^2-3}{3t^2+1}$. Hence, find $\frac{d^2y}{dx^2}$.

b. Show that $\frac{dy}{dx}$ can be expressed as $\frac{dy}{dx} = \frac{1}{3} - \frac{10}{3(3t^2+1)}$. Hence, deduce that

$-3 < \frac{dy}{dx} < \frac{1}{3}$.

END OF QUESTION PAPER

1. Express $\frac{3x^2-5}{(x-3)(x^2+2)}$ in partial fractions.

SOLUTION

$$\frac{3x^2-5}{(x-3)(x^2+2)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2}$$

$$= \frac{A(x^2+2)+(Bx+C)(x-3)}{(x-3)(x^2+2)}$$

$$3x^2-5 = A(x^2+2)+(Bx+C)(x-3)$$

$\boldsymbol{When\ x = 3}$

$$3(3)^2-5 = A[(3)^2+2]+[B(3)+C][(3)-3]$$

$$22 = 11A$$

$$A = 2$$

$\boldsymbol{When\ x = 0}$

$$3(0)^2-5 = (2)[(0)^2+2]+[B(0)+C][(0)-3]$$

$$-5 = 4-3C$$

$$3C = 9$$

$$C = 3$$

$\boldsymbol{When\ x = 1}$

$$3(1)^2-5 = (2)[(1)^2+2]+[B(1)+(3)][(1)-3]$$

$$-2 = 6-2[B+3]$$

$$2[B+3] = 8$$

$$B+3 = 4$$

$$B = 1$$

$$\frac{3x^2-5}{(x-3)(x^2+2)} = \frac{2}{x-3} + \frac{x+3}{x^2+2}$$

2. Solve the equation $\cos\theta + \cos 5\theta = 2\cos 3\theta \;\; for\; 0 \le \theta \le \pi$. Give your answers in terms of π.

SOLUTION

Factor Formulae

a) $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

b) $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

c) $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

d) $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$0 \le \theta \le \pi$

$\cos\theta + \cos 5\theta = 2\cos 3\theta$

$\cos 5\theta + \cos\theta = 2\cos 3\theta$

$2\cos\left(\frac{5\theta+\theta}{2}\right)\cos\left(\frac{5\theta-\theta}{2}\right) = 2\cos 3\theta$

$2\cos 3\theta\cos 2\theta = 2\cos 3\theta$

$2\cos 3\theta\cos 2\theta - 2\cos 3\theta = 0$

$\cos 3\theta\,[2\cos 2\theta - 2] = 0$

$\cos 3\theta = 0$

$0 \le \theta \le \pi$

$0 \le 3\theta \le 3\pi$

$3\theta = \frac{\pi}{2}, 2\pi - \frac{\pi}{2}, 2\pi + \frac{\pi}{2}$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

$2\cos 2\theta - 2 = 0$

$\cos 2\theta = 1$

$0 \le 2\theta \le 2\pi$

$2\theta = 0, \pi$

$\theta = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

3. Evaluate the following limits:

a. $\lim_{x\to 2} \frac{x^3-8}{x^2-2x}$

b. $\lim_{x\to\infty} \sqrt{\frac{5x+7}{6x-5}}$

SOLUTION

a. $\lim_{x\to 2} \frac{x^3-8}{x^2-2x} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x(x-2)}$

$$= \lim_{x\to 2} \frac{(x^2+2x+4)}{x}$$

$$= \frac{2^2+2(2)+4}{2}$$

$$= 6$$

b. $\lim_{x\to\infty} \sqrt{\frac{5x+7}{6x-5}} = \lim_{x\to\infty} \sqrt{\frac{\frac{5x}{x}+\frac{7}{x}}{\frac{6x}{x}-\frac{5}{x}}}$

$$= \lim_{x\to\infty} \sqrt{\frac{5+\frac{7}{x}}{6-\frac{5}{x}}}$$

$$= \sqrt{\frac{5+0}{6-0}}$$

$$= \sqrt{\frac{5}{6}}$$

4. Given $y = e^{-2x}\sin 3x$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Hence, show that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$.

SOLUTION

$$y = e^{-2x}\sin 3x$$

$u = e^{-2x}$ $\qquad v = \sin 3x$

$u' = -2e^{-2x}$ $\qquad v' = 3\cos 3x$

$$\boldsymbol{\frac{dy}{dx} = uv' + vu'}$$

$$= (e^{-2x})(3\cos 3x) + (\sin 3x)(-2e^{-2x})$$

$$= 3\,e^{-2x}\cos 3x - 2e^{-2x}\sin 3x$$

$$\boldsymbol{\frac{dy}{dx}} = e^{-2x}(3\cos 3x - 2\sin 3x)$$

$u = e^{-2x}$ $\qquad v = 3\cos 3x - 2\sin 3x$

$u' = -2e^{-2x}$ $\qquad v' = -9\sin 3x - 6\cos 3x$

$$\boldsymbol{\frac{d^2y}{dx^2} = uv' + vu'}$$

$$= (e^{-2x})(-9\sin 3x - 6\cos 3x) + (3\cos 3x - 2\sin 3x)(-2e^{-2x})$$

$$= (e^{-2x})[-9\sin 3x - 6\cos 3x - 2(3\cos 3x - 2\sin 3x)]$$

$$= (e^{-2x})[-9\sin 3x - 6\cos 3x - 6\cos 3x + 4\sin 3x]$$

$$= (e^{-2x})[-5\sin 3x - 12\cos 3x]$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y$$

$$= (e^{-2x})[-5\sin 3x - 12\cos 3x] + 4[3\,e^{-2x}\cos 3x - 2e^{-2x}\sin 3x] + 13[e^{-2x}\sin 3x]$$

$$= -5\,e^{-2x}\sin 3x - 12\cos e^{-2x}3x + 12\,e^{-2x}\cos 3x - 8e^{-2x}\sin 3x + 13e^{-2x}\sin 3x$$

$$= 0$$

5. Given the polynomial $P(x) = x^2 - 4$ and $Q(x) = \alpha x^4 + x^3 + 2x^2 + \beta x + 28$.

   a. Find all zeroes of $P(x)$.

   b. When $Q(x)$ is divided by $P(x)$, the remainder is $14x + 52$. Use the remainder theorem to find the values of α and β.

   c. Using the values of α and β obtained from part $5(b)$, find the remainder when $2Q(x) + x$ is divided by $P(x)$.

SOLUTION

$P(x) = x^2 - 4$

$Q(x) = \alpha x^4 + x^3 + 2x^2 + \beta x + 28$

(a) When $P(x) = 0$

$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm\sqrt{4}$

$= \pm 2$

$Therefore, zeros\ are -2, 2$

(b) $D(x) = P(x) = x^2 - 4 = (x + 2)(x - 2)$

$Q(x) = \alpha x^4 + x^3 + 2x^2 + \beta x + 28$

$R(x) = 14x + 52$

Remainder Theorem

When a polynomial P(x) is divided by (x – a), then the remainder is P(a)

$\alpha x^4 + x^3 + 2x^2 + \beta x + 28 = Q(x)(x^2 - 4) + (14x + 52)$

$\alpha x^4 + x^3 + 2x^2 + \beta x + 28 = Q(x)(x + 2)(x - 2) + (14x + 52)$

$When\ x = 2$

$Q(2) = R(2)$

$\alpha(2)^4 + (2)^3 + 2(2)^2 + \beta(2) + 28 = [14(2) + 52]$

$16\alpha + 16 + 2\beta + 28 = [28 + 52]$

$16\alpha + 2\beta = 36$

$8\alpha + \beta = 18$ .... (1)

$When\ x = -2$

$Q(-2) = R(-2)$

$\alpha(-2)^4 + (-2)^3 + 2(-2)^2 + \beta(-2) + 28 = [14(-2) + 52]$

$16\alpha - 8 + 8 - 2\beta + 28 = [-28 + 52]$

$16\alpha - 2\beta + 28 = 24$

$16\alpha - 2\beta = -4$

$8\alpha - \beta = -2$ .... (2)

$(1) + (2)$

$16\alpha = 16$

$\alpha = 1$

$8 - \beta = -2$

$\beta = 10$

$\therefore \alpha = 1, \quad \beta = 10$

(c) $Q(x) = x^4 + x^3 + 2x^2 + 10x + 28$

$P(x) = 2Q(x) + x$

$D(x) = x^2 - 4$

$(x^4 + x^3 + 2x^2 + 10x + 28) = [Q(x)(x + 2)(x - 2) + (14x + 52)]$

$2(x^4 + x^3 + 2x^2 + 10x + 28) = 2[Q(x)(x + 2)(x - 2) + (14x + 52)]$

$2[x^4 + x^3 + 2x^2 + 10x + 28] + x = 2[Q(x)(x + 2)(x - 2) + (14x + 52)] + x$

$2[x^4 + x^3 + 2x^2 + 10x + 28] + x = 2Q(x)(x + 2)(x - 2) + 2(14x + 52) + x$

$R(x) = 2(14x + 52) + x$

$= 28x + 104 + x$

$= 29x + 104$

6. Express $\cos\theta + \sqrt{2}\sin\theta$ in the form $R\sin(\theta+\alpha)$, where $R > 0$ and α is an acute angle.

   Hence,

   a. Solve the equation $\cos\theta + \sqrt{2}\sin\theta = \frac{\sqrt{3}}{2}$ by giving all solutions between $0°$ and $360°$.

   b. Show the greatest value of $\frac{1}{\cos\theta+\sqrt{2}\sin\theta+5}$ is $\frac{5+\sqrt{3}}{22}$.

SOLUTION

$\cos\theta + \sqrt{2}\sin\theta = R\sin(\theta+\alpha)$

$\cos\theta + \sqrt{2}\sin\theta = R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

$\cos\theta + \sqrt{2}\sin\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$\cos\theta + \sqrt{2}\sin\theta = R\cos\theta\sin\alpha + R\sin\theta\cos\alpha$

$R\cos\theta\sin\alpha = \cos\theta$

$R\sin\alpha = 1$ .... (1)

$R\sin\theta\cos\alpha = \sqrt{2}\sin\theta$

$R\cos\alpha = \sqrt{2}$ .... (2)

$(1)^2 + (2)^2$

$(R^2\sin^2\alpha) + (R^2\cos^2\alpha) = 1^2 + \left(\sqrt{2}\right)^2$

$R^2(\sin^2\alpha + \cos^2\alpha) = 3$

$\sin^2\alpha + \cos^2\alpha = 1$

$R^2 = 3$

$R = \sqrt{3}$

(1) ÷ (2)

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{2}}$$

$$\tan\alpha = \frac{1}{\sqrt{2}}$$

$$\alpha = 35.26°$$

$$\cos\theta + \sqrt{2}\sin\theta = \sqrt{3}\sin(\theta + 35.26°)$$

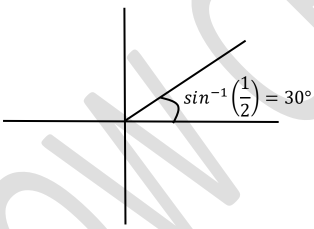
(a) $\cos\theta + \sqrt{2}\sin\theta = \frac{\sqrt{3}}{2}$ $0° < \theta < 360°$

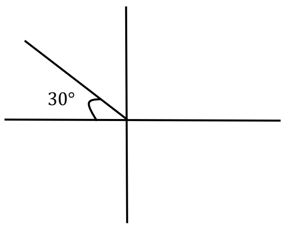
$$\sqrt{3}\sin(\theta + 35.26°) = \frac{\sqrt{3}}{2}$$

$$\sin(\theta + 35.26°) = \frac{1}{2}$$

$$0° + 35.26° < \theta + 35.26° < 360° + 35.26°$$

$$35.26° < \theta + 35.26° < 395.26°$$





$$\theta + 35.26° = 180° - 30°;\ 360° + 30°$$

$$\theta + 35.26° = 150°;\ 390°$$

$$\theta = 150° - 35.26° = 150°;\ 390° - 35.26°$$

$$\theta = 114.7°, 354.74°$$

(b) Show the greatest value of $\frac{1}{\cos\theta+\sqrt{2}\sin\theta+5}$ is $\frac{5+\sqrt{3}}{22}$

$$\cos\theta + \sqrt{2}\sin\theta = \sqrt{3}\sin(\theta + 35.26°)$$

$$-1 \le \sin(\theta + 35.26°) \le 1$$

$$-\sqrt{3} \le \sqrt{3}\sin(\theta + 35.26°) \le \sqrt{3}$$

$$-\sqrt{3} \le \cos\theta + \sqrt{2}\sin\theta \le \sqrt{3}$$

$$-\sqrt{3} + 5 \le \cos\theta + \sqrt{2}\sin\theta + 5 \le \sqrt{3} + 5$$

$$\frac{1}{\sqrt{3}+5} \le \frac{1}{\cos\theta + \sqrt{2}\sin\theta + 5} \le \frac{1}{-\sqrt{3}+5}$$

$$\frac{1}{\sqrt{3}+5} \le \frac{1}{\cos\theta + \sqrt{2}\sin\theta + 5} \le \frac{1}{5-\sqrt{3}}$$

The greatest value of $\frac{1}{\cos\theta+\sqrt{2}\sin\theta+5}$:

$$\frac{1}{5-\sqrt{3}} = \frac{1\left(5+\sqrt{3}\right)}{\left(5-\sqrt{3}\right)\left(5+\sqrt{3}\right)}$$

$$= \frac{5+\sqrt{3}}{25-3}$$

$$= \frac{5+\sqrt{3}}{22}$$

7. State $the\ conditions\ for\ continuity\ of\ f(x)\ at\ x = a.$

   a. By using the conditions for continuity of $f(x)$ at $x = a$, find the values of m and n such that

$$f(x) = \begin{cases} n - 2\cos x, & x < 0 \\ 2 + mx^2, & 0 \le x < 2 \\ m - x, & x \ge 2 \end{cases}$$

   is continuous on the interval $(-\infty, \infty)$.

   b. $If\ m = -2\ and\ n = 4$, determine whether $f(x)$ is differentiable at $x = 2$ or not.

SOLUTION

$The\ conditions\ for\ continuity\ of\ f(x)\ at\ x = a$

i. $f(a)\ is\ defined$
ii. $\lim_{x \to a} f(x)\ \ is\ exist$
iii. $\lim_{x \to a} f(x) = f(a)$

(a) $f(x) = \begin{cases} n - 2\cos x, & x < 0 \\ 2 + mx^2, & 0 \le x < 2 \\ m - x, & x \ge 2 \end{cases}$

$f(x) is\ continuous\ at\ x = 0\ and\ x = 2\ as\ well.$

$When\ x = 0$

$$\lim_{x \to 0^-} (n - 2\cos x) = \lim_{x \to 0^+} (2 + mx^2)$$

$$n - 2\cos(0) = 2 + m(0)^2$$

$$n - 2 = 2$$

$$n = 4$$

$When\ x = 2$

$$\lim_{x \to 2^-} (2 + mx^2) = \lim_{x \to 2^+} (m - x)$$

$$2 + m(2^2) = m - 2$$

$$3m + 4 = 0$$

$$m = -\frac{4}{3}$$

$$\therefore m = -\frac{4}{3}, \quad n = 4$$

(b) $If\ m = -2\ and\ n = 4$

$$f(x) = \begin{cases} 4 - 2\cos x, & x < 0 \\ 2 - 2x^2, & 0 \le x < 2 \\ -2 - x, & x \ge 2 \end{cases}$$

$At\ x = 2$

$$f'(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{(2 - 2x^2) - (-2 - x)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{2 - 2x^2 + 2 + x}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{-2x^2 + x + 4}{x - 2}$$

$$= \frac{-2(2)^2 + (2) + 4}{2 - 2}$$

$= +\infty$ ((Undefined)

$Since\ f'(2^-)\ is\ undefined, therefore\ f(x) is\ differentiable\ at\ x = 2.$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a^-) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$$

$$f'(a^+) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

8. A curve with equation $x^2 - 3y^2 = ae^{y-2x} + by - 6$, where a and b are constants, passes through the point $(1, 2)$.

   a. Given $\frac{dy}{dx} = 1$ at $(1, 2)$, determine the values a and b.

   b. Evaluate $\frac{d^2y}{dx^2}$ at $(1, 2)$.

SOLUTION

$$x^2 - 3y^2 = ae^{y-2x} + by - 6$$

$\boldsymbol{At\ the\ point\ (1, 2)}$

$$1^2 - 3(2)^2 = ae^{2-2(1)} + b(2) - 6$$

$$1 - 12 = a + 2b - 6$$

$$a + 2b = -5 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$\boldsymbol{At\ the\ point\ (1, 2), \frac{dy}{dx} = 1}$

$$x^2 - 3y^2 = ae^{y-2x} + by - 6$$

$$2x - 6y\frac{dy}{dx} = ae^{y-2x}\frac{d}{dx}(y - 2x) + b\frac{dy}{dx}$$

$$2x - 6y\frac{dy}{dx} = (ae^{y-2x})\left[\frac{dy}{dx} - 2\right] + b\frac{dy}{dx}$$

$$2(1) - 6(2)(1) = \left(ae^{(2)-2(1)}\right)[1 - 2] + b(1)$$

$$-10 = -a + b$$

$$a - b = 10 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

(1) − (2)

$$3b = -15$$

$$b = -5$$

$$a + 5 = 10$$

$$a = 5$$

$$\therefore a = 5, \qquad b = -5$$

(b) Evaluate $\frac{d^2y}{dx^2}$ at $(1, 2)$

$$x^2 - 3y^2 = 5e^{y-2x} - 5y - 6$$

$$2x - 6y\frac{dy}{dx} = (5e^{y-2x})\left[\frac{dy}{dx} - 2\right] - 5\frac{dy}{dx}$$

$$2x - 6y\frac{dy}{dx} = 5e^{y-2x}\frac{dy}{dx} - 10e^{y-2x} - 5\frac{dy}{dx}$$

$$2 - \left[6y\frac{d^2y}{dx^2} + 6\left(\frac{dy}{dx}\right)^2\right] = \left[5e^{y-2x}\frac{d^2y}{dx^2} + 5e^{y-2x}\frac{dy}{dx}\frac{d}{dx}(y-2x)\right] - 10e^{y-2x}\frac{d}{dx}(y-2x) - 5\frac{d^2y}{dx^2}$$

$$2 - \left[6y\frac{d^2y}{dx^2} + 6\left(\frac{dy}{dx}\right)^2\right] = \left[5e^{y-2x}\frac{d^2y}{dx^2} + 5e^{y-2x}\frac{dy}{dx}\left(\frac{dy}{dx} - 2\right)\right] - 10e^{y-2x}\left(\frac{dy}{dx} - 2\right) - 5\frac{d^2y}{dx^2}$$

At the point $\boldsymbol{(1, 2), \frac{dy}{dx} = 1}$

$$2 - \left[6y\frac{d^2y}{dx^2} + 6\left(\frac{dy}{dx}\right)^2\right] = \left[5e^{y-2x}\frac{d^2y}{dx^2} + 5e^{y-2x}\frac{dy}{dx}\left(\frac{dy}{dx} - 2\right)\right] - 10e^{y-2x}\left(\frac{dy}{dx} - 2\right) - 5\frac{d^2y}{dx^2}$$

$$2 - \left[6(2)\frac{d^2y}{dx^2} + 6(1)^2\right] = \left[5e^{2-2(1)}\frac{d^2y}{dx^2} + 5e^{2-2(1)}(1)((1) - 2)\right] - 10e^{2-2(1)}((1) - 2) - 5\frac{d^2y}{dx^2}$$

$$2 - \left[12\frac{d^2y}{dx^2} + 6\right] = \left[5\frac{d^2y}{dx^2} - 5\right] + 10 - 5\frac{d^2y}{dx^2}$$

$$2 - 12\frac{d^2y}{dx^2} - 6 = 5\frac{d^2y}{dx^2} - 5 + 10 - 5\frac{d^2y}{dx^2}$$

$$5\frac{d^2y}{dx^2} + 12\frac{d^2y}{dx^2} - 5\frac{d^2y}{dx^2} = 2 + 5 - 10 - 6$$

$$12\frac{d^2y}{dx^2} = -9$$

$$\frac{d^2y}{dx^2} = -\frac{9}{12}$$

$$\frac{d^2y}{dx^2} = -\frac{3}{4}$$

9. The function f is defined by $f(x) = \frac{\ln(x-1)}{x-1}$ for $x > 1$.

   a. By considering the first and second derivatives of $f(x)$, show that there is only one maximum point on the graph $y = f(x)$.

   b. Use the result obtained in part 9(a) to state the exact coordinates of the maximum point.

   c. Find the x-coordinate of the function f when $\frac{d^2y}{dx^2} = 0$.

SOLUTION

(a) $f(x) = \frac{\ln(x-1)}{x-1}, \quad for\ x > 1$

$u = \ln(x-1) \qquad v = x - 1$

$u' = \frac{1}{x-1}\frac{d}{dx}(x-1) \qquad v' = 1$

$= \dfrac{1}{x-1}$

$$f'(x) = \frac{vu' - uv'}{v^2}$$

$$= \frac{(x-1)\left(\frac{1}{x-1}\right) - [\ln(x-1)](1)}{(x-1)^2}$$

$$= \frac{1 - \ln(x-1)}{(x-1)^2}$$

$Let\ f'(x) = 0$

$$\frac{1 - \ln(x-1)}{(x-1)^2} = 0$$

$1 - \ln(x-1) = 0$

$\ln(x-1) = 1$

$x - 1 = e^1$

$x = e + 1$

$\boldsymbol{log_a b = c \iff b = a^c}$

$\boldsymbol{ln\, b = c \iff b = e^c}$

$$f'(x) = \frac{1 - \ln(x-1)}{(x-1)^2}$$

$u = 1 - \ln(x-1)$ $\qquad$ $v = (x-1)^2$

$u' = -\frac{1}{x-1}$ $\qquad$ $v' = 2(x-1)$

$$f''(x) = \frac{(x-1)^2\left(-\frac{1}{x-1}\right) - [1 - \ln(x-1)]2(x-1)}{[(x-1)^2]^2}$$

$$= \frac{[-(x-1)] - [1 - \ln(x-1)](2x-2)}{(x-1)^4}$$

$When\ x = e + 1$

$$f''(x) = \frac{\big(-(e+1-1)\big) - [1 - \ln(e+1-1)][(2(e+1)-2)]}{(e+1-1)^4}$$

$$= \frac{(-e) - [1 - \ln(e)](2e)}{(e)^4}$$

$$= \frac{(-e)}{(e)^4}$$

$$= -\frac{1}{e^3} < 0 \ \ (Maximum)$$

(b) Use the result obtained in part 9(a) to state the exact coordinates of the maximum point.

$$f(x) = \frac{\ln(x-1)}{x-1}$$

$When\ x = e + 1$

$$f(x) = \frac{\ln(e+1-1)}{e+1-1}$$

$$= \frac{\ln e}{e}$$

$$= \frac{1}{e}$$

$\therefore$ *the exact coordinates of the maximum point*: $\left(e+1, \frac{1}{e}\right)$

(c) Find the x-coordinate of the function f when $\frac{d^2y}{dx^2} = 0$

$$When\ \frac{d^2y}{dx^2} = 0$$

$$\frac{[-(x-1)] - [1 - \ln(x-1)](2x-2)}{(x-1)^4} = 0$$

$$[-(x-1)] - [1 - \ln(x-1)](2x-2) = 0$$

$$[-(x-1)] - 2(x-1)[1 - \ln(x-1)] = 0$$

$$(x-1)[-1 - 2(1 - \ln(x-1))] = 0$$

$$(x-1)[-1 - 2 + 2\ln(x-1)] = 0$$

$$(x-1)[-3 + 2\ln(x-1)] = 0$$

$$(x-1) = 0 \qquad -3 + 2\ln(x-1) = 0$$

$$x = 1 \qquad 2\ln(x-1) = 3$$

$$\ln(x-1) = \frac{3}{2}$$

$$x - 1 = e^{\frac{3}{2}}$$

$$x = e^{\frac{3}{2}} + 1$$

$$Since\ x \neq 1, therfore\ x = e^{\frac{3}{2}} + 1$$

10. A curve is defined by the parametric equations $x = 3t - \frac{1}{t}$ and $y = t + \frac{3}{t}$, where $t \neq 0$.

a. Show that $\frac{dy}{dx} = \frac{t^2-3}{3t^2+1}$. Hence, find $\frac{d^2y}{dx^2}$.

b. Show that $\frac{dy}{dx}$ can be expressed as $\frac{dy}{dx} = \frac{1}{3} - \frac{10}{3(3t^2+1)}$. Hence, deduce that

$-3 < \frac{dy}{dx} < \frac{1}{3}$.

SOLUTION

$x = 3t - \frac{1}{t}$ and $y = t + \frac{3}{t}$

(a) Show that $\frac{dy}{dx} = \frac{t^2-3}{3t^2+1}$. Hence, find $\frac{d^2y}{dx^2}$

$$x = 3t - \frac{1}{t}$$

$$\frac{dx}{dt} = 3 + \frac{1}{t^2}$$

$$= \frac{3t^2 + 1}{t^2}$$

$$y = t + \frac{3}{t}$$

$$\frac{dy}{dt} = 1 - \frac{3}{t^2}$$

$$= \frac{t^2 - 3}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{t^2 - 3}{t^2} \cdot \frac{t^2}{3t^2 + 1}$$

$$= \frac{t^2 - 3}{3t^2 + 1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{t^2 - 3}{3t^2 + 1}$$

$u = t^2 - 3$ $\qquad$ $v = 3t^2 + 1$

$u' = 2t$ $\qquad$ $v' = 6t$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{vu' - uv'}{v^2}$$

$$= \frac{(3t^2 + 1)(2t) - (t^2 - 3)(6t)}{(3t^2 + 1)^2}$$

$$= \frac{6t^3 + 2t - 6t^3 + 18t}{(3t^2 + 1)^2}$$

$$= \frac{20t}{(3t^2 + 1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$= \frac{20t}{(3t^2 + 1)^2} \cdot \frac{t^2}{3t^2 + 1}$$

$$= \frac{20t^3}{(3t^2 + 1)^3}$$

(b) Show that $\frac{dy}{dx}$ can be expressed as $\frac{dy}{dx} = \frac{1}{3} - \frac{10}{3(3t^2+1)}$. Hence, deduce that

$$-3 < \frac{dy}{dx} < \frac{1}{3}.$$

$$\frac{dy}{dx} = \frac{t^2 - 3}{3t^2 + 1}$$

$$\begin{array}{r|l} & \frac{1}{3} \\ 3t^2 + 1 & t^2 - 3 \\ & t^2 + \frac{1}{3} \\ \hline & -\frac{10}{3} \end{array}$$

$$\frac{dy}{dx} = \frac{1}{3} + \frac{-\frac{10}{3}}{3t^2 + 1}$$

$$\frac{1}{3} - \frac{10}{3(3t^2 + 1)}$$

$$t^2 > 0$$

$$3t^2 > 0$$

$$3t^2 + 1 > 0 + 1$$

$$3t^2 + 1 > 1$$

$$3(3t^2 + 1) > 3$$

$$3 < 3(3t^2 + 1) < \infty$$

$$0 < \frac{1}{3(3t^2 + 1)} < \frac{1}{3}$$

$$0 < \frac{10}{3(3t^2 + 1)} < \frac{10}{3}$$

$$-\frac{10}{3} < -\frac{10}{3(3t^2 + 1)} < 0$$

$$\frac{1}{3} - \frac{10}{3} < \frac{1}{3} - \frac{10}{3(3t^2 + 1)} < \frac{1}{3} + 0$$

$$-3 < \frac{1}{3} - \frac{10}{3(3t^2 + 1)} < \frac{1}{3}$$

$$-3 < \frac{dy}{dx} < \frac{1}{3}$$