KOLEJ MATRIKULASI KEDAH
KEMENTERIAN PENDIDIKAN MALAYSIA

QS025/2

Matriculation Programme Examination

Semester 2

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Chow Choon Wooi

1. A sample of positive integers is arranged in ascending order as follow:

$$3x + 2, 40, 4x, 2y, 59, 3y - 9$$

If the mean and median of the sample are 49 and 47 respectively, determine the values of x and y. Hence, rewrite the sample in ascending order.

2. Let A and B be two events where $P(A') = 0.7, P(B) = 0.4$ and $P(A \cup B) = 0.6$.

Determine $P(A \cap B)$ and then evaluate $P(A|B)$.

Hence, state with reason whether A and B are independent events.

3. The number of text messages received by Rosnaida during a fixed time interval is distributed with a mean of 6 messages per hour.

   a) Find the probability that Rosnaida will receive exactly 8 messages between 16:00 and 18:00 on a particular day.

   b) It is known that Rosnaida has received at least 10 messages between 16:00 and 18:00 on a particular day, find the probability that she received 13 text messages during that time interval.

4. A probability distribution for discrete random variable X is as shown in the table below.

X	*-2*	*-1*	*0*	*1*	*2*	*3*
P(X=x)	*p*	*0.1*	*0.3*	*q*	*0.2*	*p+q*

Where p and q are constants. If E(X)=0.65, determine the values of p and q. Hence, calculate the standard deviation of *X*.

5. The probability distribution function of a discrete random variable X is given as:

$$f(x) = \begin{cases} \frac{x}{17} & , \quad x = 1,2,3 \\ \frac{x}{34} & , \quad x = 4,5,6,7 \\ 0 & , \quad Otherwise \end{cases}$$

   a) Calculate $P(2 \leq X \leq 5)$

   b) Determine the value of Var(X).

   Hence, calculate the standard deviation of $Y = (\sqrt{5}X - 1)$.

6. The frequency distribution for 80 employees at a supermarket according to their daily wage class is as shown below.

Class Boundary of Daily Wage (RM)	Frequency
15 – 20	4
20 – 25	12
25 – 30	20
30 – 35	32
35 – 40	8
40 – 45	4

a) Find the median and standard deviation of the sample.

b) Calculate and interpret the Pearson's coefficient of skewness for the data.

c) Determine the daily wage k where 80% of the workers earn at most k ringgit per day.

7. A box contains 12 cups of similar size and shape but in different colours. There are 5 blue cups, 4 red cups and 3 yellow cups. A rack can only take up 6 cups. In how many ways can:

   a) Any 6 cups be placed on the rack?

   b) An equal number of colored cups that could be placed on the rack?

   c) An equal number of colored cups that could be placed on the rackwith cups of the same colour being side by side?

   d) An equal number of yellow and red cups be placed on the rack?

8. A survey was implemented on 400 students at a private university in order to collect information on the popular choice of minor subjects (Language, Statistics and Information Technology) by students of various major of study (Medicine, Engineering and Economics). The following table describes the data collected from the survey.

Minor \ Major	Medicine	Engineering	Economic	Total
Language	30	80	30	140
Statistics	10	30	10	50
Information Technology	60	120	30	210
Total	100	230	70	400

If a student from this group is selected at random, what is the probability that he:

a) Is either majoring in Medicine or doing a minor in Information Technology?

b) Is a non-Medicine student who does a minor in Language?

c) Chooses a minor in Statistics knowing that he is an Economics student?

d) Is neigher an Engineering student who does a minor Statistics nor is he an Economics student who does a minor in Language?

9. In every delivery of cupcakes to a particular restaurant, 30% will be returned due to not favoured by cupcakes lovers.

   a) Suppose 20 of the cupcakes are randomly selected from a delivery.What is the probability that at most 5 will be returned?

   b) Suppose the restaurant will be holding an event which requires an order of 200 cupcakes from the same supplier.

      i. Approximate the probability that between 56 and 62 of the cupcake will be returned.

      ii. If the probability of observing less that n number of cupcakes among thos delivered which are returned is 0.992, use the normal approximation to determine the value of n.

10. Continous random variable X has a density probability function given by

$$f(x) = \begin{cases} ax \quad , \quad 0 \le x \le 1 \\ \dfrac{a}{3}(4-x) \quad , \quad 1 \le x \le 4 \\ 0 \quad , \quad otherwise \end{cases}$$

where a is a constant.

   a) Find the value of a.

   b) Find the $E(X)$ and $Var\ (2-3X)$.

   c) Evaluate $P(X - E(X) < a)$.

   d) Estimate the median.

END OF QUESTION PAPER

1. A sample of positive integers is arranged in ascending order as follow:

$$3x+2, 40, 4x, 2y, 59, 3y-9$$

If the mean and median of the sample are 49 and 47 respectively, determine the values of x and y. Hence, rewrite the sample in ascending order.

SOLUTION

$3x+2, 40, 4x, 2y, 59, 3y-9$

Mean $= 49$

$$Mean = \frac{3x+2+40+4x+2y+59+3y-9}{6}$$

$$49 = \frac{7x+5y+92}{6}$$

$$7x+5y+92 = 294$$

$$7x+5y = 202 \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Median $= 47$

$3x+2, 40, 4x, 2y, 59, 3y-9$

$$\frac{4x+2y}{2} = 47$$

$$4x+2y = 94$$

$$2x+y = 47$$

$$y = 47-2x \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

Subtitute (2) *into* (1)

$$7x+5(47-2x) = 202$$

$$7x+235-10x = 202$$

$$33 = 3x$$

$$x = 11$$

$$y = 47-2(11) = 25$$

$\therefore x = 11; y = 25$

$3x+2, 40, 4x, 2y, 59, 3y-9$

$3(11)+2, 40, 4(11), 2(25), 59, 3(25)-9$

$35, 40, 44, 50, 59, 66$

2. Let A and B be two events where $P(A') = 0.7, P(B) = 0.4\ and\ P(A \cup B) = 0.6$.

   Determine $P(A \cap B)$ and then evaluate $P(A|B)$.

   Hence, state with reason whether A and B are independent events.

SOLUTION

$P(A') = 0.7, \quad P(B) = 0.4 \qquad P(A \cup B) = 0.6$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A \cup B) - P(A) - P(B)$

$= 0.6 - (1 - 0.7) - 0.4$

$= 0.1$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{0.1}{0.4}$$

$$= 0.25$$

Since $P(A|B) \neq P(A)$, therefore A and B are not independent events.

Independent Events

$$P(A \cap B) = P(A).P(B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A|B).P(B)$$

Conclusion:

If A and B are independent events, then $P(A|B) = P(A)$

3. The number of text messages received by Rosnaida during a fixed time interval is distributed with a mean of 6 messages per hour.
   a) Find the probability that Rosnaida will receive exactly 8 messages between 16:00 and 18:00 on a particular day.
   b) It is known that Rosnaida has received at least 10 messages between 16:00 and 18:00 on a particular day, find the probability that she received 13 text messages during that time interval.

SOLUTION

$\lambda = 6\ messages\ per\ hour$

a) $Between\ 16:00\ and\ 18:00$ ➔ 2 hours ➔ $\lambda = 12\ messages\ per\ two\ hour$

$X \sim P_o(12)$

$$P(X = 8) = P(X \geq 8) - P(X \geq 9)$$
$$= 0.9105 - 0.8450$$
$$= 0.0655$$

b) $$P(X = 13 | X \geq 10) = \frac{P(X = 13 \cap X \geq 10)}{P(X \geq 10)}$$
$$= \frac{P(X = 13)}{P(X \geq 10)}$$
$$= \frac{P(X \geq 13) - P(X \geq 14)}{P(X \geq 10)}$$
$$= \frac{0.4240 - 0.3185}{0.7576}$$
$$= 0.1393$$

$P(X = 13 \cap X \geq 10) = P(X = 13)$

$If\ A = \{13\}, B = \{10, 11, 12, 13, \dots\}$

$P(A \cap B) = \{13\}$

4. A probability distribution for discrete random variable X is as shown in the table below.

X	*-2*	*-1*	*0*	*1*	*2*	*3*
P(X=x)	*p*	*0.1*	*0.3*	*q*	*0.2*	*p+q*

Where p and q are constants. If E(X)=0.65, determine the values of p and q. Hence, calculate the standard deviation of *X*.

SOLUTION

$Discreate\ Random\ Variable\ X: \sum P(X=x) = 1$

$p + 0.1 + 0.3 + q + 0.2 + p + q = 1$

$2p + 2q = 0.4$

$p + q = 0.2$ ……………... (1)

Given that E(x)=0.65

$E(x) = \sum x\, P(X=x)$

$(-2)(p) + (-1)(0.1) + (0)(0.3) + (1)(q) + (2)(0.2) + (3)(p+q) = 0.65$

$-2p - 0.1 + 0 + q + 0.4 + 3p + 3q = 0.65$

$p + 4q = 0.35$ ……………... (2)

(2) – (1)

$3q = 0.15$

$q = 0.05$

$From$ (1):

$p = 0.2 - q$

$= 0.2 - 0.05$

$= 0.15$

X	-2	-1	0	1	2	3
$P(X=x)$	0.15	0.1	0.3	0.05	0.2	0.2

$Var(x) = E(X^2) - [E(x)]^2$

$E(X^2) = \sum x^2 P(X = x)$

$E(X^2) = (-2)^2(0.15) + (-1)^2(0.1) + (0)^2(0.3) + (1)^2(0.05) + (2)^2(0.2) + (3)^2(0.2)$

$= 0.6 + 0.1 + 0 + 0.05 + 0.8 + 1.8$

$= 3.35$

$Var(x) = 3.35 - (0.65)^2$

$= 2.9275$

$Standard\ Deviation\ of\ x, \sigma = \sqrt{var(x)}$

$= \sqrt{2.9275}$

$= 1.711$

5. The probability distribution function of a discrete random variable X is given as:

$$f(x)=\begin{cases}\frac{x}{17} & , \quad x=1,2,3\\ \frac{x}{34} & , \quad x=4,5,6,7\\ 0 & , \quad Otherwise\end{cases}$$

a) Calculate $P(2 \le X < 5)$

b) Determine the value of Var(X).

Hence, calculate the standard deviation of $Y = \left(\sqrt{5}X - 1\right)$.

SOLUTION

$$f(x)=\begin{cases}\frac{x}{17} & , \quad x=1,2,3\\ \frac{x}{34} & , \quad x=4,5,6,7\\ 0 & , \quad Otherwise\end{cases}$$

a) $P(2 \le X < 5) = P(X=2) + P(X=3) + P(X=4)$

$$= \frac{2}{17} + \frac{3}{17} + \frac{4}{34}$$

$$= \frac{14}{34}$$

$$= \frac{7}{17}$$

b) $Var(x) = E(X^2) - [E(x)]^2$

$$E(x) = (1)\left(\frac{1}{17}\right) + (2)\left(\frac{2}{17}\right) + (3)\left(\frac{3}{17}\right) + (4)\left(\frac{4}{34}\right) + (5)\left(\frac{5}{34}\right) + (6)\left(\frac{6}{34}\right) + (7)\left(\frac{7}{34}\right)$$

$$= \frac{1}{17} + \frac{4}{17} + \frac{9}{17} + \frac{16}{34} + \frac{25}{34} + \frac{36}{34} + \frac{49}{34}$$

$$= \frac{154}{34}$$

$$= \frac{77}{17}$$

$$E(x^2) = (1)^2\left(\frac{1}{17}\right) + (2)^2\left(\frac{2}{17}\right) + (3)^2\left(\frac{3}{17}\right) + (4)^2\left(\frac{4}{34}\right) + (5)^2\left(\frac{5}{34}\right) + (6)^2\left(\frac{6}{34}\right) + (7)^2\left(\frac{7}{34}\right)$$

$$= \frac{1}{17} + \frac{8}{17} + \frac{27}{17} + \frac{64}{34} + \frac{125}{34} + \frac{216}{34} + \frac{343}{34}$$

$$= \frac{820}{34} = \frac{410}{17}$$

$Var(x) = E(X^2) - [E(x)]^2$

$$= \frac{410}{17} - \left(\frac{77}{17}\right)^2$$
$$= \frac{1041}{289}$$

When $Y = (\sqrt{5}X - 1)$

$$Var(Y) = Var(\sqrt{5}X - 1)$$
$$= (\sqrt{5})^2 Var(X)$$
$$= 5\left(\frac{1041}{289}\right)$$
$$= \frac{5205}{289}$$

1. $Var(a) = 0$
2. $Var(ax + b) = a^2 var(x)$

$$standard\ deviation\ of\ Y, \sigma = \sqrt{\frac{5205}{289}}$$
$$= 4.244$$

6. The frequency distribution for 80 employees at a supermarket according to their daily wage class is as shown below.

Class Boundary of Daily Wage (RM)	Frequency
15 – 20	4
20 – 25	12
25 – 30	20
30 – 35	32
35 – 40	8
40 – 45	4

a) Find the median and standard deviation of the sample.
b) Calculate and interpret the Pearson's coefficient of skewness for the data.
c) Determine the daily wage k where 80% of the workers earn at most k ringgit per day.

SOLUTION

Class Boundary of Daily Wage (RM)	Mid Point X	Frequency (f)	Cummulative Frequency (F)	fx	fx^2
15 – 20	17.5	4	4	70	1225
20 – 25	22.5	12	16	270	6075
25 – 30	27.5	20	36	550	15125
30 – 35	32.5	32	68	1040	33800
35 – 40	37.5	8	76	300	11250
40 – 45	42.5	4	80	170	7225
			Total	2400	74700

(a) $Median = L_k + \left(\frac{\frac{n}{2} - F_{k-1}}{f_k}\right) C$

$L_k = 30; \quad n = 80; \quad F_{k-1} = 36; \quad f_k = 32; \quad C = 5$

$$Median = 30 + \left(\frac{\frac{80}{2} - 36}{32}\right)5$$

$$= 30.625$$

$$Standard\ Deviation\ \sigma = \sqrt{\frac{\sum fx^2 - \frac{(\sum fx)^2}{n}}{n-1}}$$

$$= \sqrt{\frac{74700 - \frac{(2400)^2}{80}}{80-1}}$$

$$= 5.846$$

(b) Pearson's coefficient of skewness, $S_k = \frac{3(Mean - Media\ \ )}{Standard\ Deviation}$

$$Mean, \bar{x} = \frac{\sum fx}{\sum f}$$

$$= \frac{2400}{80}$$

$$= 30$$

$$S_k = \frac{3(Mean - Median)}{Standard\ Deviation}$$

$$= \frac{3(30 - 30.625)}{5.846}$$

$$= -0.321$$

$Therefore, the\ distribution\ is\ skewed\ to\ the\ left\ (negatively\ skewed)$

(c) $k = P_{80}$

$$P_k = L_k + \left(\frac{\frac{kn}{100} - F_{k-1}}{f_k}\right)C$$

Class Boundary of Daily Wage (RM)	Mid Point X	Frequency (f)	Cummulative Frequency (F)
15 – 20	17.5	4	4
20 – 25	22.5	12	16
25 – 30	27.5	20	36
30 – 35	32.5	32	68
35 – 40	37.5	8	76
40 – 45	42.5	4	80

$$P_k = L_k + \left(\frac{\frac{kn}{100} - F_{k-1}}{f_k}\right)C$$

$L_k = 30; \quad k = 80; \quad n = 80; \quad F_{k-1} = 36; \quad f_k = 32; \quad C = 5$

$$P_{80} = 30 + \left(\frac{\frac{(80)(80)}{100} - 36}{32}\right)5$$

$$= 34.375$$

$\therefore k = 34.375$

7. A box contains 12 cups of similar size and shape but in different colours. There are 5 blue cups, 4 red cups and 3 yellow cups. A rack can only take up 6 cups. In how many ways can:
   a) Any 6 cups be placed on the rack?
   b) An equal number of colored cups that could be placed on the rack?
   c) An equal number of colored cups that could be placed on the rack with cups of the same colour being side by side?
   d) An equal number of yellow and red cups be placed on the rack?

SOLUTION

5B, 4R, 3Y: B, B, B, B, B, R, R, R, R, Y, Y, Y

(a)

Blue	Red	Yellow	Permutation		
5	1	0	$\frac{6!}{5!\,1!\,0!}$	=	6
5	0	1	$\frac{6!}{5!\,0!\,1!}$	=	6
4	2	0	$\frac{6!}{4!\,2!\,0!}$	=	15
4	1	1	$\frac{6!}{4!\,1!\,1!}$	=	30
4	0	2	$\frac{6!}{4!\,0!\,2!}$	=	15
3	3	0	$\frac{6!}{3!\,3!\,0!}$	=	20
3	2	1	$\frac{6!}{3!\,2!\,1!}$	=	60
3	1	2	$\frac{6!}{3!\,1!\,2!}$	=	60
3	0	3	$\frac{6!}{3!\,0!\,3!}$	=	20
2	4	0	$\frac{6!}{2!\,4!\,0!}$	=	15
2	3	1	$\frac{6!}{2!\,3!\,1!}$	=	60
2	2	2	$\frac{6!}{2!\,2!\,2!}$	=	90
2	1	3	$\frac{6!}{2!\,1!\,3!}$	=	60
1	4	1	$\frac{6!}{1!\,4!\,1!}$	=	30

1	3	2	$\frac{6!}{1!\,3!\,2!}$	=	60
1	2	3	$\frac{6!}{1!\,2!\,3!}$	=	60
0	4	2	$\frac{6!}{0!\,4!\,2!}$	=	15
0	3	3	$\frac{6!}{0!\,3!\,3!}$	=	20
				Total	642

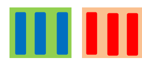
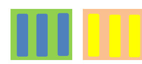
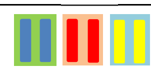
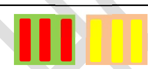
$$The\ number\ of\ arrangement = 642$$

(b) An equal number of colored cups that could be placed on the rack?

Blue	Red	Yellow	Permutation		
3	3	0	$\frac{6!}{3!\,3!\,0!}$	=	20
3	0	3	$\frac{6!}{3!\,0!\,3!}$	=	20
2	2	2	$\frac{6!}{2!\,2!\,2!}$	=	90
0	3	3	$\frac{6!}{0!\,3!\,3!}$	=	20
				Total	150

$$The\ number\ of\ arrangement = 150$$

(c) An equal number of colored cups that could be placed on the rack with cups of the same colour being side by side?

Blue	Red	Yellow		Permutation		
3	3	0	 $\frac{3!}{3!}$ $\frac{3!}{3!}$ ⏟ $2!$	$\frac{3!}{3!}x\frac{3!}{3!}x2!$	=	2
3	0	3	 $\frac{3!}{3!}$ $\frac{3!}{3!}$ ⏟ $2!$	$\frac{3!}{3!}x\frac{3!}{3!}x2!$	=	2
2	2	2	 $\frac{2!}{2!}$ $\frac{2!}{2!}$ $\frac{2!}{2!}$ ⏟ $3!$	$\frac{2!}{2!}x\frac{2!}{2!}x\frac{2!}{2!}x3!$	=	6
0	3	3	 $\frac{3!}{3!}$ $\frac{3!}{3!}$ ⏟ $2!$	$\frac{3!}{3!}x\frac{3!}{3!}x2!$	=	2
					Total	12

$$The\ number\ of\ arrangement = 12$$

(d) An equal number of yellow and red cups be placed on the rack?

Blue	Red	Yellow	Permutation		
4	1	1	$\frac{6!}{4!\,1!\,1!}$	=	30
2	2	2	$\frac{6!}{2!\,2!\,2!}$	=	90
0	3	3	$\frac{6!}{0!\,3!\,3!}$	=	20
				Total	140

$$The\ number\ of\ arrangement = 140$$

8. A survey was implemented on 400 students at a private university in order to collect information on the popular choice of minor subjects (Language, Statistics and Information Technology) by students of various major of study (Medicine, Engineering and Economics). The following table describes the data collected from the survey.

Major / Minor	Medicine (Me)	Engineering (En)	Economic (Ec)	Total
Language (L)	30	80	30	140
Statistics (S)	10	30	10	50
Information Technology (I)	60	120	30	210
Total	100	230	70	400

If a student from this group is selected at random, what is the probability that he:

a) Is either majoring in Medicine or doing a minor in Information Technology?
b) Is a non-Medicine student who does a minor in Language?
c) Chooses a minor in Statistics knowing that he is an Economics student?
d) Is neither an Engineering student who does a minor Statistics nor is an Economics student who does a minor in Language?

SOLUTION

$$\boldsymbol{Me - Major\ Medicine}$$
$$\boldsymbol{En - Major\ Engineering}$$
$$\boldsymbol{Ec - Major\ Economic}$$
$$\boldsymbol{L - Minor\ Language}$$
$$\boldsymbol{S - Minor\ Statistics}$$
$$\boldsymbol{I - Minor\ Information\ Technology}$$

(a) $P(Me \cup I) = P(Me) + P(I) - P(Me \cap I)$

$$= \frac{100}{400} + \frac{210}{400} - \frac{60}{400}$$
$$= \frac{250}{400}$$
$$= \frac{5}{8}$$

(b) $P(Me' \cap L) = P(L) - P(Me \cap L)$

$$= \frac{140}{400} - \frac{30}{400}$$

$$= \frac{110}{400}$$

$$= \frac{11}{40}$$

(c) $P(S|Ec) = \frac{P(S \cap Ec)}{P(Ec)}$

$$= \frac{\frac{10}{400}}{\frac{70}{400}}$$

$$= \frac{10}{400} x \frac{400}{70}$$

$$= \frac{1}{7}$$

(d) $P[(En \cap S)' \cap (Ec \cap L)')] = P[(En \cap S) \cup (Ec \cap L)]'$

$$= 1 - P[(En \cap S) \cup (Ec \cap L)]$$

$$= 1 - \left(\frac{30}{400} + \frac{30}{400}\right)$$

$$= \frac{17}{20}$$

9. In every delivery of cupcakes to a particular restaurant, 30% will be returned due to not favoured by cupcakes lovers.
   a) Suppose 20 of the cupcakes are randomly selected from a delivery. What is the probability that at most 5 will be returned?
   b) Suppose the restaurant will be holding an event which requires an order of 200 cupcakes from the same supplier.
      i. Approximate the probability that between 56 and 62 of the cupcake will be returned.
      ii. If the probability of observing less that n number of cupcakes among those delivered which are returned is 0.992, use the normal approximation to determine the value of n.

SOLUTION

a) $p = 0.3, \quad n = 20$

$X \sim B(20, 0.3)$

$P(X \leq 5) = 1 - P(X \geq 6)$

$= 1 - 05836$

$= 0.4164$

b) $n = 200; \quad p = 0.3$

$X \sim B(200, 0.3)$

$Since\ np = 60 > 5, \ \ nq = 140 > 5, therefore$

$X \sim B(n, p) \implies X \sim N[np, npq]$

$X \sim B(200, 0.3) \implies X \sim N[(200)(0.3), (200)(0.3)(0.7)]$

$X \sim B(200, 0.3) \implies X \sim N(60,42)$

i.

$X \sim B(200, 0.3)$	$\implies$	$X \sim N(60,42)$	$\implies$	$Z \sim N(0,1)$
$P(56 < X < 62)$	$=$	$P(56.5 \leq X \leq 61.5)$	$=$	$P\left(\frac{56.5 - 60}{\sqrt{42}} \leq Z \leq \frac{61.5 - 60}{\sqrt{42}}\right)$
			$=$	$P(-0.54 \leq Z \leq 0.23)$
			$=$	$1 - P(Z \geq 0.54) - P(Z \geq 0.23)$
			$=$	$1 - 0.2946 - 0.4090$
			$=$	0.2964

ii. $P(X < n) = 0.992$

$P(X < n - 0.5) = 0.992$

$P\left(Z < \frac{(n - 0.5) - 60}{\sqrt{42}}\right) = 0.992$

$P\left(Z < \frac{n - 60.5}{\sqrt{42}}\right) = 0.992$

$Let\ k = \frac{n - 60.5}{\sqrt{42}}$

$P(Z < k) = 0.992$

$P(Z \geq k) = 1 - 0.992$

$P(Z \geq k) = 0.008$

$k = 2.41$

$2.41 = \frac{n - 60.5}{\sqrt{42}}$

$n = 76.1$

$n = 76$

10. Continous random variable X has a probability density function given by

$$f(x)=\begin{cases} ax & , \quad 0\le x\le 1 \\ \frac{a}{3}(4-x) & , \quad 1\le x\le 4 \\ 0 & , \quad otherwise \end{cases}$$

where a is a constant.

a) Find the value of a.

b) Find the $E(X)$ and $Var\ (2-3X)$.

c) Evaluate $P(X-E(X)<a)$.

d) Estimate the median.

SOLUTION

$$f(x)=\begin{cases} ax & , \quad 0\le x\le 1 \\ \frac{a}{3}(4-x) & , \quad 1\le x\le 4 \\ 0 & , \quad otherwise \end{cases}$$

a) For probability density function ➔ $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} 0dx + \int_{0}^{1} ax\,dx + \int_{1}^{4} \frac{a}{3}(4-x)dx + \int_{4}^{\infty} 0dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + \frac{a}{3}\left[4x-\frac{x^2}{2}\right]_1^4 = 1$$

$$a\left[\left(\frac{1^2}{2}\right)-\left(\frac{0^2}{2}\right)\right] + \frac{a}{3}\left[\left(4(4)-\frac{4^2}{2}\right)-\left(4(1)-\frac{1^2}{2}\right)\right] = 1$$

$$\frac{1}{2}a + \frac{a}{3}\left[8-\frac{7}{2}\right] = 1$$

$$\frac{1}{2}a + \frac{a}{3}\left[\frac{9}{2}\right] = 1$$

$$\frac{1}{2}a + \frac{3}{2}a = 1$$

$$2a = 1$$

$$a = \frac{1}{2}$$

b) $E(x) = \int_{-\infty}^{\infty} x.f(x)dx$

$$f(x) = \begin{cases} \frac{1}{2}x & , \quad 0 \le x \le 1 \\ \frac{1}{6}(4-x) & , \quad 1 \le x \le 4 \\ 0 & , \quad otherwise \end{cases}$$

$$E(x) = \int_{-\infty}^{0} x(0)dx + \int_{0}^{1} x\left(\frac{1}{2}x\right) dx + \int_{1}^{4} x\left[\frac{1}{6}(4-x)\right]dx + \int_{4}^{\infty} x(0)dx$$

$$= \frac{1}{2}\int_{0}^{1} x^2\, dx + \frac{1}{6}\int_{1}^{4} 4x - x^2 dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^1 + \frac{1}{6}\left[\frac{4x^2}{2} - \frac{x^3}{3}\right]_1^4$$

$$= \frac{1}{2}\left[\left(\frac{1^3}{3}\right) - \left(\frac{0^3}{3}\right)\right] + \frac{1}{6}\left[\left(\frac{4(4)^2}{2} - \frac{(4)^3}{3}\right) - \left(\frac{4(1)^2}{2} - \frac{(1)^3}{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{6}\left[\left(32 - \frac{64}{3}\right) - \left(2 - \frac{1}{3}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{6}\left[\frac{32}{3} - \frac{5}{3}\right]$$

$$= \frac{1}{6} + \frac{1}{6}\left[\frac{27}{3}\right]$$

$$= \frac{10}{6}$$

$$= \frac{5}{3}$$

$$E(x^2) = \int_{-\infty}^{0} x^2(0)dx + \int_{0}^{1} x^2\left(\frac{1}{2}x\right) dx + \int_{1}^{4} x^2\left[\frac{1}{6}(4-x)\right]dx + \int_{4}^{\infty} x^2(0)dx$$

$$= \frac{1}{2}\int_{0}^{1} x^3\, dx + \frac{1}{6}\int_{1}^{4} 4x^2 - x^3 dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 + \frac{1}{6}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_1^4$$

$$= \frac{1}{2}\left[\left(\frac{1^4}{4}\right) - \left(\frac{0^4}{4}\right)\right] + \frac{1}{6}\left[\left(\frac{4(4)^3}{3} - \frac{4^4}{4}\right) - \left(\frac{4(1)^3}{3} - \frac{1^4}{4}\right)\right]$$

$$= \frac{1}{8} + \frac{1}{6}\left[\left(\frac{256}{3} - \frac{256}{4}\right) - \left(\frac{4}{3} - \frac{1}{4}\right)\right]$$

$$= \frac{1}{8} + \frac{1}{6}\left[\left(\frac{1024 - 768}{12}\right) - \left(\frac{16 - 3}{12}\right)\right]$$

$$= \frac{1}{8} + \frac{1}{6}\left[\left(\frac{256}{12}\right) - \left(\frac{13}{12}\right)\right]$$

$$= \frac{1}{8} + \frac{1}{6}\left(\frac{243}{12}\right)$$

$$= \frac{1}{8} + \frac{81}{24}$$

$$= \frac{84}{24}$$

$$= \frac{7}{2}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{7}{2} - \left(\frac{5}{3}\right)^2$$

$$= \frac{7}{2} - \frac{25}{9}$$

$$= \frac{63 - 50}{18}$$

$$= \frac{13}{18}$$

$$Var\,(2 - 3X) = (-3)^2 Var(x)$$

$$= 9\left(\frac{13}{18}\right)$$

$$= \frac{13}{2}$$

$Var(ax + b) = a^2 var(x)$

c) $P(X - E(X) < a) = P(X - \frac{5}{3} < \frac{1}{2})$

$$= P(X < \frac{1}{2} + \frac{5}{3})$$

$$= P\left(X < \frac{13}{6}\right)$$

$$= \int_{-\infty}^{\frac{13}{6}} f(x)dx$$

$$= \int_{-\infty}^{0} 0dx + \frac{1}{2}\int_{0}^{1} xdx + \frac{1}{6}\int_{1}^{\frac{13}{6}} 4 - x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{6}\left[4x - \frac{x^2}{2}\right]_1^{\frac{13}{6}}$$

$$= \frac{1}{2}\left[\left(\frac{1^2}{2}\right) - \left(\frac{0^2}{2}\right)\right] + \frac{1}{6}\left[\left(4\left(\frac{13}{6}\right) - \frac{\left(\frac{13}{6}\right)^2}{2}\right) - \left(4(1) - \frac{1^2}{2}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{6}\left[\left(\frac{52}{6} - \frac{169}{72}\right) - \left(4 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{6}\left[\left(\frac{624 - 169}{72}\right) - \left(\frac{7}{2}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{6}\left[\frac{455}{72} - \frac{7}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{6}\left[\frac{455 - 252}{72}\right]$$

$$= \frac{1}{4} + \frac{1}{6}\left[\frac{203}{72}\right]$$

$$= \frac{1}{4} + \frac{203}{432}$$

$$= \frac{108 + 203}{432}$$

$$= \frac{311}{432}$$

d) Median, m ➔ $F(m) = 0.5$

$$P(X \le m) = 0.5$$

$$\int_{-\infty}^{m} f(x)dx = 0.5$$

$$\int_{-\infty}^{0} 0\,dx + \frac{1}{2}\int_{0}^{1} xdx + \frac{1}{6}\int_{1}^{m} (4 - x)dx = 0.5$$

$$\frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{6}\left[4x - \frac{x^2}{2}\right]_1^m = 0.5$$

$$\frac{1}{4} + \frac{1}{6}\left[\left(4m - \frac{m^2}{2}\right) - \left(4(1) - \frac{1^2}{2}\right)\right] = 0.5$$

$$\frac{1}{4}+\frac{1}{6}\left[\left(4m-\frac{m^2}{2}\right)-\left(4-\frac{1}{2}\right)\right]=0.5$$

$$\frac{1}{4}+\frac{1}{6}\left[\left(\frac{8m-m^2}{2}\right)-\left(\frac{7}{2}\right)\right]=0.5$$

$$\frac{1}{4}+\frac{8m-m^2-7}{12}=0.5$$

$$\frac{8m-m^2-7}{12}=0.5-\frac{1}{4}$$

$$\frac{8m-m^2-7}{12}=\frac{1}{4}$$

$$8m-m^2-7=3$$

$$m^2-8m+10=0$$

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$m=\frac{-(-8)\pm\sqrt{(-8)^2-4(1)(10)}}{2(1)}$$

$$m=\frac{8\pm\sqrt{64-40}}{2}$$

$$m=\frac{8\pm\sqrt{24}}{2}$$

$$m=6.45 \qquad or \qquad m=1.55$$

$Since\ 1\le m\le 4, therefore\ m=1.55$