KEMENTERIAN PENDIDIKAN MALAYSIA

QS 015/2

Matriculation Programme Examination

Semester I

Session 2016/2017

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1. Express $\frac{x^2}{x^2-2x-3}$ in partial fractions form.

2. Evaluate the following limits, if exist.

    a. $\lim_{x \to 2} \frac{x-2}{x^4-16}$

    b. $\lim_{x \to \infty} \frac{(2-x)(x-1)}{(x-3)^2}$

3. Show that $\frac{sin^2 x}{1-\cos x} = 1 + \cos x$. Hence, solve $\frac{sin^2 x}{1-\cos x} = \cos 2x$ for $0° \leq x \leq 360°$.

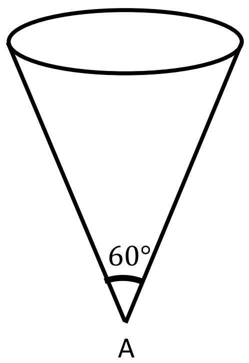
4. Consider a function $f(x) = \frac{1}{2-\sqrt{x}}$.

    a. Find $\lim_{x \to \infty} f(x)$ and state the equation of horizontal asymptote for f.

    b. By using the first principle of derivative, find $f'(x)$.

5. (a) Use the derivative to find the maximum area of a rectangle that can be inscribed in a semicircle of radius 10cm.

   (b) A cone-shaped tank as shown below.



Water flows through a hole A at rate of 6 cm^3 per second. Find the rate of change in height of the water when the volume of water in the cone is $24\pi cm^3$

6. (a) Polynomial $P(x)$ has a ramindér 3 when divided by $(x+3)$. Find the remainder of $P(x)+2$ when divided by $(x + 3)$.

   (b) Polynomial $P_1(x) = x^3 + ax^2 - 5bx - 7$ has a factor $(x - 1)$ and remainder R_1 when divided by $(x + 1)$, while a polynomial $P_2(x) = x^3 - ax^2 + bx + 6$ has a remainder R_2 when divided by $(x - 1)$. Find the value of the constants a and b if $R_1 + R_2 = 5$. Hence, obtain the zeroes for $P_1(x)$.

7. Consider a function $f(x) = \sqrt{3}\cos 2x + 2\sin 2x$.

   a. Express f in the form of $R\cos(2x - \alpha)$ for $R > 0$, $0° \leq \alpha \leq 90°$ and α to the nearest minute. State the maximum and minimum values of f.

   b. Hence, solve $\sqrt{3}\cos 2x + 2\sin 2x = -\sqrt{2}$ for $0° \leq x \leq 180°$. Give your answer to the nearest minute.

8. The parametric equations of a curve is given by

$$x = e^{2t+1}, \qquad y = e^{-(2t-1)}$$

   (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $t = 1$.

   (b) Given $z = x^2 - xy$. Express z in terms of t and find $\frac{dz}{dt}$. Hence, duduce the set value of t such that $\frac{dz}{dt}$ is positive.

9. (a) Given $f(x) = \frac{2|x|}{x} + 5x$. Compute $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$. Is f continuous at x=0? Give your reason.

   (b) The continuous function g is defined by

$$f(x) = \begin{cases} \sqrt{5 - x}, & x < a \\ 3x - 1, & x \geq a \end{cases}$$

   Find the value of a.

10. By writing $tan\, x$ in terms of $sin\, x$ and $cos\, x$, show that

$$\frac{d}{dx}(\tan x) = sec^2 x.$$

   a. If $y = tan\, x$, find $\frac{d^2y}{dx^2}$ in terms of y. Hence, determine the range of value of x such that $\frac{d^2y}{dx^2} > 0$ for $0 < x < \pi$.

   b. If $y = tan\,(x + y)$, find $\frac{dy}{dx}$ in terms of x and y.

   Hence, show that $\frac{dy}{dx} = -cosec^2 2\alpha$ when $x = y = \alpha$.

END OF QUESTION PAPER

1. Express $\frac{x^2}{x^2-2x-3}$ in partial fractions form.

SOLUTION

$$\frac{x^2}{x^2-2x-3}$$

Improper Fraction

$$\begin{array}{r} 1 \\ x^2-2x-3 \,\big)\, \overline{x^2 } \\ \underline{x^2-2x-3} \\ 2x+3 \end{array}$$

$$\frac{x^2}{x^2-2x-3} = 1 + \frac{2x+3}{x^2-2x-3}$$

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

$$\frac{2x+3}{x^2-2x-3} = \frac{2x+3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$\frac{2x+3}{(x-3)(x+1)} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

$$2x+3 = A(x+1) + B(x-3)$$

$When\ x = -1$

$$2(-1)+3 = A\big((-1)+1\big) + B\big((-1)-3\big)$$

$$1 = -4B$$

$$B = -\frac{1}{4}$$

$When\ x = 3$

$$2(3)+3 = A\big((3)+1\big) + B\big((3)-3\big)$$

$$9 = 4A$$

$$A = \frac{9}{4}$$

$$\frac{2x+3}{x^2-2x-3} = \frac{9}{4(x-3)} - \frac{1}{4(x+1)}$$

$$\frac{x^2}{x^2-2x-3} = 1 + \frac{2x+3}{x^2-2x-3}$$

$$= 1 + \frac{9}{4(x-3)} - \frac{1}{4(x+1)}$$

2. Evaluate the following limits, if exist.

a. $\lim_{x \to 2} \frac{x-2}{x^4-16}$

b. $\lim_{x \to \infty} \frac{(2-x)(x-1)}{(x-3)^2}$

SOLUTION

a. $\lim_{x \to 2} \frac{x-2}{x^4-16} = \lim_{x \to 2} \frac{x-2}{(x^2+4)(x^2-4)}$

$$= \lim_{x \to 2} \frac{x-2}{(x^2+4)(x+2)(x-2)}$$

$$= \lim_{x \to 2} \frac{1}{(x^2+4)(x+2)}$$

$$= \frac{1}{(2^2+4)(2+2)}$$

$$= \frac{1}{32}$$

b. $\lim_{x \to \infty} \frac{(2-x)(x-1)}{(x-3)^2} = \lim_{x \to \infty} \frac{2x-2-x^2+x}{x^2-6x+9}$

$$= \lim_{x \to \infty} \frac{-x^2+3x-2}{x^2-6x+9}$$

$$= \lim_{x \to \infty} \frac{\frac{-x^2+3x-2}{x^2}}{\frac{x^2-6x+9}{x^2}}$$

$$= \lim_{x \to \infty} \frac{-1+\frac{3}{x}-\frac{2}{x^2}}{1-\frac{6}{x}+\frac{9}{x^2}}$$

$$= \frac{-1+0+0}{1}$$

$$= -1$$

3. Show that $\frac{sin^2 x}{1-\cos x} = 1 + \cos x$. Hence, solve $\frac{sin^2 x}{1-\cos x} = \cos 2x$ for $0° \leq x \leq 360°$.

SOLUTION

$$\frac{sin^2 x}{1-\cos x} = \frac{1-\cos^2 x}{1-\cos x}$$

$$= \frac{(1-\cos x)(1+\cos x)}{1-\cos x}$$

$$= 1 + \cos x$$

$$\frac{sin^2 x}{1-\cos x} = \cos 2x$$

$$1 + \cos x = 2\,cos^2 x - 1$$

$$2\,cos^2 x - \cos x - 2 = 0$$

$$Let\ y = \cos x$$

$$2\,y - y - 2 = 0$$

$$y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-2)}}{2(2)}$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

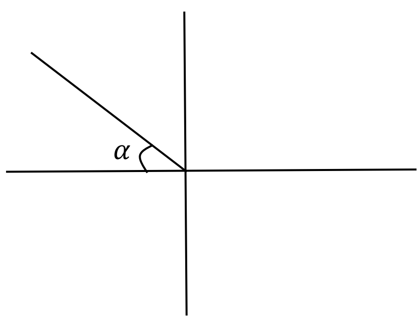
$$= \frac{1 \pm \sqrt{17}}{4}$$

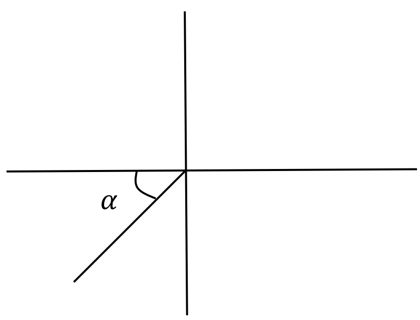
$$= 1.2808, -0.7808$$

$$\cos x = 1.2808\,, -0.7808$$

Since $-1 \leq \cos x \leq 1$

$$\cos x = -0.7808$$





$\alpha = cos^{-1}0.7808$

$= 38.67°$

Given that $0° \leq x \leq 360°$

$x = 180° - 38.67, 180° + 38.67$

$= 141.33°, 218.67°$

4. Consider a function $f(x) = \frac{1}{2-\sqrt{x}}$.

   a. Find $\lim_{x\to\infty} f(x)$ and state the equation of horizontal asymptote for f.

   b. By using the first principal of derivative, find $f'(x)$.

SOLUTION

$$f(x) = \frac{1}{2-\sqrt{x}}$$

a. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{2-\sqrt{x}}$

$$= 0$$

$\therefore f(x) = 0 \; is \; horizontal \; asymptote.$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

b. $f'(x) = \lim_{h\to 0} \frac{1}{h}[f(x+h) - f(x)]$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{2-\sqrt{x+h}} - \frac{1}{2-\sqrt{x}}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(2-\sqrt{x}) - (2-\sqrt{x+h})}{(2-\sqrt{x+h})(2-\sqrt{x})}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{2-\sqrt{x}-2+\sqrt{x+h}}{(2-\sqrt{x+h})(2-\sqrt{x})}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-\sqrt{x}+\sqrt{x+h}}{(2-\sqrt{x+h})(2-\sqrt{x})}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sqrt{x+h}-\sqrt{x}}{(2-\sqrt{x+h})(2-\sqrt{x})}\right]\left[\frac{(\sqrt{x+h}+\sqrt{x})}{\sqrt{x+h}+\sqrt{x}}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h) - (\sqrt{x}\sqrt{x+h}) + (\sqrt{x}\sqrt{x+h}) - (x)}{(2-\sqrt{x+h})(2-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h}{(2-\sqrt{x+h})(2-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}\right]$$

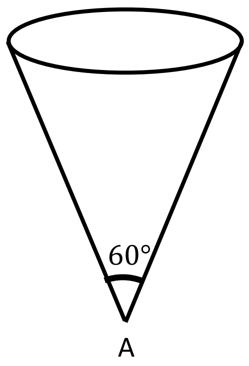
$$= \lim_{h\to 0} \left[\frac{1}{(2-\sqrt{x+h})(2-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}\right]$$

$$= \frac{1}{\left(2-\sqrt{x}\right)\left(2-\sqrt{x}\right)\left(\sqrt{x}+\sqrt{x}\right)}$$

$$= \frac{1}{2\sqrt{x}\left(2-\sqrt{x}\right)^2}$$

5. (a) Use the derivative to find the maximum area of a rectangle that can be inscribed in a semicircle of radius 10cm.

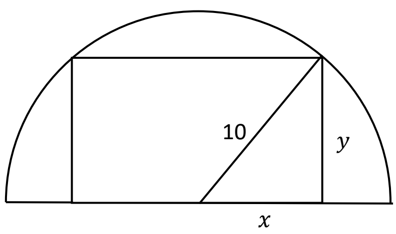
(b) A cone-shaped tank as shown below.



Water flows through a hole A at rate of 6 cm^3 per second. Find the rate of change in height of the water when the volume of water in the cone is $24\pi cm^3$

<u>SOLUTION</u>

(a)



$x^2 + y^2 = 100$

$y^2 = 100 - x^2$

$y = (100 - x^2)^{\frac{1}{2}}$

Area of rectangle

$\boldsymbol{A = 2xy}$

$A = 2x(100 - x^2)^{\frac{1}{2}}$

$u = 2x$

$u' = 2$

$v = (100 - x^2)^{\frac{1}{2}}$

$v' = \frac{1}{2}(100 - x^2)^{-\frac{1}{2}} \frac{d}{dx}(100 - x^2)$

$= \frac{1}{2(100 - x^2)^{\frac{1}{2}}}(-2x)$

$= \frac{-2x}{2(100 - x^2)^{\frac{1}{2}}}$

$= \frac{-x}{(100 - x^2)^{\frac{1}{2}}}$

$$\frac{dA}{dx} = uv' + vu'$$

$$= (2x)\left(\frac{-x}{(100 - x^2)^{\frac{1}{2}}}\right) + \left((100 - x^2)^{\frac{1}{2}}\right)(2)$$

$$= \frac{-2x^2}{(100 - x^2)^{\frac{1}{2}}} + 2(100 - x^2)^{\frac{1}{2}}$$

$$= \frac{-2x^2 + 2(100 - x^2)^{\frac{1}{2}}(100 - x^2)^{\frac{1}{2}}}{(100 - x^2)^{\frac{1}{2}}}$$

$$= \frac{-2x^2 + 2(100 - x^2)}{(100 - x^2)^{\frac{1}{2}}}$$

$$= \frac{-2x^2 + 200 - 2x^2}{(100 - x^2)^{\frac{1}{2}}}$$

$$= \frac{200 - 4x^2}{(100 - x^2)^{\frac{1}{2}}}$$

Let $\frac{dA}{dx} = 0$

$$\frac{200-4x^2}{(100-x^2)^{\frac{1}{2}}}=0$$

$$200-4x^2=0$$

$$4x^2=200$$

$$x^2=50$$

$$x=\pm\sqrt{50}$$

Since $x \geq 0$

$$x=\sqrt{50}$$

$$\frac{dA}{dx}=\frac{-4x^2+200}{(100-x^2)^{\frac{1}{2}}}$$

$u=-4x^2+200$ $\qquad$ $v=(100-x^2)^{\frac{1}{2}}$

$u'=-8x$ $\qquad$ $v'=\frac{1}{2}(100-x^2)^{-\frac{1}{2}}\frac{d}{dx}(100-x^2)$

$$=\frac{1}{2}(100-x^2)^{-\frac{1}{2}}(-2x)$$

$$=\frac{-x}{(100-x^2)^{\frac{1}{2}}}$$

$$\frac{d^2A}{dx^2}=\frac{vu'-uv'}{v^2}$$

$$=\frac{(100-x^2)^{\frac{1}{2}}(-8x)-(-4x^2+200)\left(\frac{-x}{(100-x^2)^{\frac{1}{2}}}\right)}{\left((100-x^2)^{\frac{1}{2}}\right)^2}$$

$$=\frac{-8x(100-x^2)^{\frac{1}{2}}+\left[\frac{x(-4x^2+200)}{(100-x^2)^{\frac{1}{2}}}\right]}{\left((100-x^2)^{\frac{1}{2}}\right)^2}$$

$$= \frac{\left[\frac{-8x(100-x^2)^{\frac{1}{2}}(100-x^2)^{\frac{1}{2}} + (-4x^3+200x)}{(100-x^2)^{\frac{1}{2}}}\right]}{100-x^2}$$

$$= \left[\frac{-8x(100-x^2) + (-4x^3+200x)}{(100-x^2)^{\frac{1}{2}}}\right]\left[\frac{1}{100-x^2}\right]$$

$$= \frac{-800x + 8x^3 - 4x^3 + 200x}{(100-x^2)^{\frac{3}{2}}}$$

$$= \frac{4x^3 - 600x}{(100-x^2)^{\frac{3}{2}}}$$

$When\ x = \sqrt{50}$

$$\frac{d^2A}{dx^2} = \frac{4\left(\sqrt{50}\right)^3 - 600\sqrt{50}}{\left[100 - \left(\sqrt{50}\right)^2\right]^{\frac{3}{2}}} = -8 < 0\ (\text{max})$$

The maximum area, $A_{max} = 2x(100-x^2)^{\frac{1}{2}}$

$$= 2\sqrt{50}\left(100 - \sqrt{50}^2\right)^{\frac{1}{2}}$$

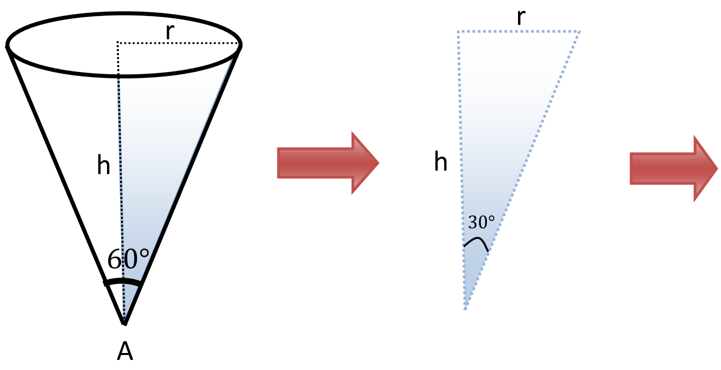
$$= 2\sqrt{50}(50)^{\frac{1}{2}}$$

$$= 2\sqrt{50}\sqrt{50}$$

$$= 2(50)$$

$$= 100cm^2$$

(b)



$$\frac{r}{h} = \tan 30°$$

$$\frac{r}{h} = \frac{1}{\sqrt{3}}$$

$$r = \frac{h}{\sqrt{3}}$$

$$\frac{dv}{dt} = -6cm^3s^{-1}$$

$$Find\ \frac{dh}{dt}\ when\ v = 24\pi cm^3$$

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$$

$$v = \frac{1}{3}\pi r^2 h$$

$$Since\ r = \frac{h}{\sqrt{3}}$$

$$v = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$$

$$= \frac{1}{3}\pi\frac{h^3}{3}$$

$$= \frac{\pi}{9}h^3$$

$$\frac{dv}{dh} = \frac{\pi h^2}{3}$$

$$\frac{dh}{dv} = \frac{3}{\pi h^2}$$

$$\frac{dh}{dt} = \frac{3}{\pi h^2} \cdot (-6)$$

$$\frac{dh}{dt} = -\frac{18}{\pi h^2}$$

$when\ v = 24\pi$

$$\frac{\pi}{9}h^3 = 24\pi$$

$$h^3 = 24\pi\ x\ \frac{9}{\pi}$$

$$= 216$$

$$h = 6$$

$$\frac{dh}{dt} = -\frac{18}{\pi 6^2}$$

$$\frac{dh}{dt} = -\frac{18}{\pi(6)^2}$$

$$= -\frac{1}{2\pi}\ cms^{-1}$$

6. (a) Polynomial $P(x)$ has a raminder 3 when divided by $(x+3)$. Find the remainder of $P(x)+2$ when divided by $(x+3)$.

(b) Polynomial $P_1(x)=x^3+ax^2-5bx-7$ has a factor $(x-1)$ and remainder R_1 when divided by $(x+1)$, while a polynomial $P_2(x)=x^3-ax^2+bx+6$ has a remainder R_2 when divided by $(x-1)$. Find the value of the constants a and b if $R_1+R_2=5$. Hence, obtain the zeroes for $P_1(x)$.

SOLUTION

(a) $P(-3)=3$

$P(x)=Q(x)D(x)+R(x)$

$P(x)=Q(x)(x+3)+3$

$P(x)+2=Q(x)(x+3)+3+2$

$P(x)+2=Q(x)(x+3)+5$

$\therefore R(x)=5\ when\ \ P(x)+2\ divided\ by\ (x+3)$

(b) $P_1(x)=x^3+ax^2-5bx-7$

$P_1(1)=0$

$(1)^3+a(1)^2-5b(1)-7=0$

$1+a-5b-7=0$

$a-5b=6$ .... (1)

$P_1(-1)=R_1$

$(-1)^3+a(-1)^2-5b(-1)-7=R_1$

$-1+a+5b-7=R_1$

$a+5b=R_1+8$

$R_1=a+5b-8$ .... (2)

$P_2(x) = x^3 - ax^2 + bx + 6$

$P_2(1) = R_2$

$(1)^3 - a(1)^2 + b(1) + 6 = R_2$

$1 - a + b + 6 = R_2$

$-a + b = R_2 - 7$

$R_2 = -a + b + 7$ ……………………… (3)

$(2) + (3)$-

$R_1 + R_2 = 6b - 1$

$Given\ that\ R_1 + R_2 = 5$

$5 = 6b - 1$

$6b = 6$

$b = 1$ ……………………… (4)

$Substitute\ (4) into\ (1)$

$a - 5(1) = 6$

$a = 11$

$\therefore a = 11, b = 1$

$P_1(x) = x^3 + 11x^2 - 5x - 7$

$= (x - 1)Q(x)$

$P_1(x) = x^3 + ax^2 - 5bx - 7$ has a factor of $(x - 1)$

$$\begin{array}{r} x^2+12x+7 \\ x-1\overline{)\,x^3+11x^2-5x-7} \\ \underline{x^3-x^2\qquad\qquad} \\ 12x^2-5x-7 \\ \underline{12x^2-12x\quad} \\ 7x-7 \\ \underline{7x-7} \\ 0 \end{array}$$

$P_1(x) = x^3 + 11x^2 - 5x - 7$

$= (x-1)(x^2 + 12x + 7)$

To find the zeroes of $P_1(x)$

$P_1(x) = 0$

$(x-1)(x^2 + 12x + 7) = 0$

$x - 1 = 0$

$x = 1$

$(x^2 + 12x + 7) = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-12 \pm \sqrt{144 - 28}}{2}$$

$$= \frac{-12 \pm \sqrt{116}}{2}$$

$$= -6 \pm \sqrt{29}$$

Hence, the zeroes of $P_1(x)$ *are* $1, -6 \pm \sqrt{29}$

7. Consider a function $f(x) = \sqrt{3}\cos 2x + 2\sin 2x$.

a. Express f in the form of $R\cos(2x - \alpha)$ for $R > 0$, $0° \leq \alpha \leq 90°$ and α to the nearest minute. State the maximum and minimum values of f.

b. Hence, solve $\sqrt{3}\cos 2x + 2\sin 2x = -\sqrt{2}$ for $0° \leq x \leq 180°$. Give your anser to the nearest minute.

SOLUTION

(a)

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

$$\cos(2x - \alpha) = \cos 2x\cos\alpha + \sin 2x\sin\alpha$$

$$f(x) = \sqrt{3}\cos 2x + 2\sin 2x$$

$$\sqrt{3}\cos 2x + 2\sin 2x = R\cos(2x - \alpha)$$

$$\sqrt{3}\cos 2x + 2\sin 2x = R[\cos 2x\cos\alpha + \sin 2x\sin\alpha]$$

$$\sqrt{3}\cos 2x + 2\sin 2x = R\cos 2x\cos\alpha + \mathrm{R}\sin 2x\sin\alpha$$

$R\cos\alpha = \sqrt{3}$ .... (1)

$R\sin\alpha = 2$ .... (2)

$(1)^2 + (2)^2$

$$\cos^2\alpha + \sin^2\alpha = 1$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = \left(\sqrt{3}\right)^2 + (2)^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 3 + 4$$

$$R^2 = 7$$

$$R = \sqrt{7}$$

$(2) \div (1)$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{2}{\sqrt{3}}$$

$$\tan\alpha = \frac{2}{\sqrt{3}}$$

$$\alpha = 49.11°$$

$$\therefore f(x) = \sqrt{7}\cos(2x - 49.11°)$$

$$Since - 1 \leq \cos(2x - 49.11°) \leq 1$$

$$-\sqrt{7} \le \sqrt{7}\cos(2x - 49.11°) \le \sqrt{7}$$

$$\therefore Maximum\ value\ of\ f = \sqrt{7}$$

$$\therefore Minimum\ value\ of\ f = -\sqrt{7}$$

(b)

$\sqrt{3}\cos 2x + 2\sin 2x = -\sqrt{2}$ for $0° \le x \le 180°$

$$\sqrt{7}\cos(2x - 49.11°) = -\sqrt{2}$$

$$\cos(2x - 49.11°) = -\frac{\sqrt{2}}{\sqrt{7}}$$

$$2x - 49.11° = 180° - 57.69°, 180° + 57.69°$$

$$2x - 49.11° = 122.31°, \qquad 237.69°$$

$$2x = 171.12°, \qquad 286.80°$$

$$x = 85.56°, \qquad 143.4°$$

8. The parametric equations of a curve is given by

$$x = e^{2t+1}, \qquad y = e^{-(2t-1)}$$

(c) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $t = 1$.

(d) Given $z = x^2 - xy$. Express z in terms of t and find $\frac{dz}{dt}$. Hence, duduce the set value of t such that $\frac{dz}{dt}$ iz positive.

SOLUTION

$$x = e^{2t+1}, \qquad y = e^{-(2t-1)}$$

$$\frac{dx}{dt} = 2e^{2t+1} \qquad \frac{dy}{dt} = -2e^{-(2t-1)}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= -2e^{-(2t-1)} \cdot \frac{1}{2e^{2t+1}}$$

$$= \frac{-2e^{-(2t-1)}}{2e^{2t+1}}$$

$$= \frac{-e^{-(2t-1)}}{e^{2t+1}}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$= -\frac{1}{e^{4t}}$$

$When\ t = 1$

$$\frac{dy}{dx} = -\frac{1}{e^{4(1)}}$$

$$= -\frac{1}{e^4}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right] \cdot \frac{dt}{dx}$$

$$= \left[\frac{d}{dt}\left(-\frac{1}{e^{4t}}\right)\right] \cdot \left[\frac{1}{2e^{2t+1}}\right]$$

$$= \left[\frac{d}{dt}(-e^{-4t})\right] \cdot \left[\frac{1}{2e^{2t+1}}\right]$$

$$= 4e^{-4t}.\left[\frac{1}{2e^{2t+1}}\right]$$

$$= \frac{4e^{-4t}}{2e^{2t+1}}$$

$$= \frac{2e^{-4t}}{e^{2t+1}}$$

$$= \frac{2}{e^{6t+1}}$$

$When\ t = 1$

$$\frac{d^2y}{dx^2} = \frac{2}{e^{6t+1}}$$

$$= \frac{2}{e^{6(1)+1}}$$

$$= \frac{2}{e^7}$$

$$z = x^2 - xy$$

$$= (e^{2t+1})^2 - (e^{2t+1})\left(e^{-(2t-1)}\right)$$

$$= e^{2(2t+1)} - e^{2t+1-2t+1}$$

$$= e^{4t+2} - e^2$$

$$(a^m)^n = a^{mn}$$

$$a^m + a^n = a^{m+n}$$

$$\frac{dz}{dt} = 4e^{4t+2}$$

$$For\ \frac{dz}{dt} > 0$$

$$4e^{4t+2} > 0$$

$$e^{4t+2} > 0$$

$$Solution\ set: \{t: t \in \mathcal{R}\}$$

9. (a) Given $f(x)=\frac{2|x|}{x}+5x$. Compute $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$. Is f continuous at x=0? Give your reason.

(b) The continuous function g is defined by

$$f(x)=\begin{cases}\sqrt{5-x}, & x<a\\ 3x-1, & x\geq a\end{cases}$$

Find the value of a.

SOLUTION

(a) $f(x)=\frac{2|x|}{x}+5x$

$$|x|=\begin{cases}x, & x\geq 0\\ -x, & x<0\end{cases}$$

$$f(x)=\begin{cases}\dfrac{2x}{x}+5x, & x>0\\ \dfrac{2(-x)}{x}+5x, & x<0\end{cases}$$

$$f(x)=\begin{cases}2+5x, & x>0\\ -2+5x, & x<0\end{cases}$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} 2+5x$$

$$=2+5(0)$$

$$=2$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} -2+5x$$

$$=-2+5(0)$$

$$=2$$

$$\lim_{x\to 0^+} f(x)\neq \lim_{x\to 0^-} f(x)$$

$$\lim_{x\to 0} f(x) \text{ does not exist}$$

$$Since \lim_{x\to 0} f(x) \text{ does not exist, therefore } f \text{ is not continuous at } x=0$$

(b) $f(x) = \begin{cases} \sqrt{5-x}, & x < a \\ 3x-1, & x \geq a \end{cases}$

$f(a) = 3(a) - 1 = 3a - 1$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} 3x - 1 = 3(a) - 1 = 3a - 1$

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} \sqrt{5-x}) = \sqrt{5-a}$

$Since\ g\ is\ continuous\ at\ x = a$ ➔ $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

$3a - 1 = \sqrt{5-a}$

$(3a-1)^2 = \left(\sqrt{5-a}\right)^2$

$9a^2 - 6a + 1 = 5 - a$

$9a^2 - 5a - 4 = 0$

$(9a+4)(a-1) = 0$

$(9a+4) = 0 \qquad (a-1) = 0$

$a = -\frac{4}{9} \qquad a = 1$

$When\ a = -\frac{4}{9},$

$$3\left(-\frac{4}{9}\right) - 1 \neq \sqrt{5 - \left(-\frac{4}{9}\right)}$$

$$\therefore a \neq -\frac{4}{9}$$

$When\ a = 1,$

$$3(1) - 1 = \sqrt{5 - (1)}$$

$\therefore a = 1$

$g\ is\ continuous\ at\ x = a$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

$3a - 1 = \sqrt{5-a}$

10. By writing $tan\ x$ in terms of $sin\ x$ and $cos\ x$, show that

$$\frac{d}{dx}(\tan x) = sec^2x.$$

a. If $y = tan\ x$, find $\frac{d^2y}{dx^2}$ in terms of y. Hence, determine the range of value of x such that $\frac{d^2y}{dx^2} > 0$ for $0 < x < \pi$.

b. If $y = tan\ (x + y)$, find $\frac{dy}{dx}$ in terms of x and y.

Hence, show that $\frac{dy}{dx} = -cosec^2 2\alpha$ when $x = y = \alpha$.

SOLUTION

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$u = \sin x \qquad v = \cos x$

$u' = \cos x \qquad v' = -\sin\ x$

$=$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin\ x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= sec^2x$$

(a) $y = tan\ x$

$$\frac{dy}{dx} = sec^2x$$

$1 + tan^2x = sec^2x$

$$= 1 + tan^2x$$

$$= 1 + y^2$$

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}$$

$$= 2y(1 + y^2)$$

$For\ \frac{d^2y}{dx^2} > 0 \qquad 0 < x < \pi$

$2y(1 + y^2) > 0$

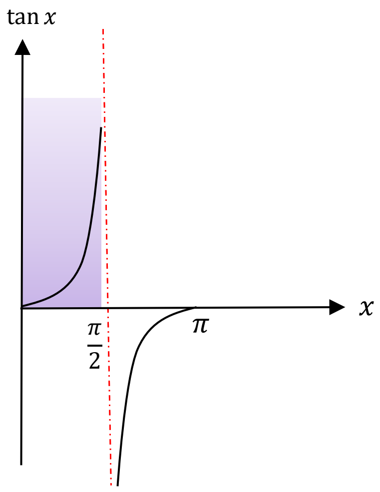
$Since\ 1 + y^2 > 0$

$2y > 0$

$y > 0$

$\tan x > 0$

$\therefore Solution\ interval\ is\ \left(0,\ \frac{\pi}{2}\right)$



(b) $y = tan\,(x + y)$

$$\frac{dy}{dx} = sec^2(x + y)\frac{d}{dx}[x + y]$$

$$\frac{dy}{dx} = sec^2(x + y)\left[1 + \frac{dy}{dx}\right]$$

$$\frac{dy}{dx} = sec^2(x + y) + sec^2(x + y)\frac{dy}{dx}$$

$$\frac{dy}{dx} - sec^2(x + y)\frac{dy}{dx} = sec^2(x + y)$$

$$\frac{dy}{dx}[1 - sec^2(x + y)] = sec^2(x + y)$$

$$\frac{dy}{dx} = \frac{sec^2(x + y)}{1 - sec^2(x + y)}$$

when $x = y = \alpha$

$$\frac{dy}{dx} = \frac{sec^2(\alpha + \alpha)}{1 - sec^2(\alpha + \alpha)}$$

$$= \frac{sec^2 2\alpha}{1 - sec^2 2\alpha}$$

$$= \frac{sec^2 2\alpha}{-tan^2 2\alpha}$$

$$= \frac{\frac{1}{cos^2 2\alpha}}{-\frac{sin^2 2\alpha}{cos^2 2\alpha}}$$

$1 + tan^2\theta = sec^2\theta$

$1 - sec^2\theta = tan^2\theta$

$$= \frac{1}{cos^2 2\alpha}\left(-\frac{cos^2 2\alpha}{sin^2 2\alpha}\right)$$

$$= -\frac{1}{sin^2 2\alpha}$$

$$= -cosec^2 2\alpha$$