

QS 025/1
Matriculation Programme
Examination
Semester I
Session 2015/2016

1. Find the equation of a circle that is passing through points (1, 2), (-1, 2) and (0, -1). Hence, determine its center.

2. Show that $\int_1^e x \ln x \, dx = \frac{1}{4}(1 + e^x)$.

3. Find y in terms of x given that $x \frac{dy}{dx} = (1 - 2x^2)y$ where $x > 0$ and $y = 1$ when $x = 1$.

4. Find the general solution of the differential equation $\frac{dy}{dx} + y \cot x = 2 \sin x$.

5. Express $\frac{1-4x}{3+x-2x^2}$ in partial fractions and hence, find the exact value of $\int_0^1 \frac{1-4x}{3+x-2x^2} dx$.

6. (a) Given $f_1(x) = 2x$ and $f_2(x) = -\ln x$.
 - i. Without using curve sketching, show that $y = f_1(x)$ and $y = f_2(x)$ intersect on the interval of [0.1, 1].
 - ii. Use Newton-Raphson's method to estimate the intersection point of $y = f_1(x)$ and $y = f_2(x)$, with the initial value $x_1 = 1$. Iterate until $|f(x_n)| < 0.005$. Give your answer correct to three decimal places.

- b) By using the trapezoidal rule, find the approximate value for $\int_0^1 x\sqrt{x+1} \, dx$ when $n = 4$, correct to four decimal places.

7. (a) If $\mathbf{p} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{q} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, show that
$$|\mathbf{p} \times \mathbf{q}|^2 = |\mathbf{p}|^2 |\mathbf{q}|^2 - (\mathbf{p} \cdot \mathbf{q})^2$$

- (b) Given a triangle ABC with $\overrightarrow{AB} = 2\mathbf{a}$ and $\overrightarrow{AC} = 3\mathbf{b}$. Use the result in part (a), show that the area of the triangle is $3\sqrt{|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2}$. Hence, deduce the area of the triangle if $\mathbf{a} = \mathbf{p}$ and $\mathbf{b} = \mathbf{q}$

8. Given a line $l: x = 2 - t, y = -3 + 4t, z = -5 - 3t$, and two planes $\pi_1: 2x - y + 7z = 53$ and $\pi_2: 3x + y + z = 1$. Find
- The point of intersection between the line l and the plane π_1 .
 - The acute angle between the line l and the plane π_1 .
 - The acute angle between planes π_1 and π_2 .
9. (a) Find the equation in standard form of an ellipse which passes through the point $(-1, 6)$ and having foci at $(-5, 2)$ and $(3, 2)$.
- (b) From the result obtained in part (a), sketch the graph of the ellipse.
10. (a) Sketch and shade the region R bounded by the curve $y = \sqrt{x}$, line $y = 2 - x$ and y -axis. Hence, find the area of the region R .
- (b) If R_1 is a region bounded by the curve $y = \sqrt{x}$, line $y = 2 - x$ and x -axis, deduce the ratio of $R: R_1$.
- (c) Find the volume of the solid generated when the region R is rotated through 360° about the x -axis.

END OF QUESTION PAPER

1. Find the equation of a circle that is passing through points (1, 2), (-1, 2) and (0, -1). Hence, determine its center.

SOLUTION

General Equation of circle:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

At (1, 2)

$$1^2 + 2^2 + 2g(1) + 2f(2) + c = 0$$

$$2g + 4f + c = -5 \quad \dots\dots\dots (1)$$

At (-1, 2)

$$(-1)^2 + 2^2 + 2g(-1) + 2f(2) + c = 0$$

$$-2g + 4f + c = -5 \quad \dots\dots\dots (2)$$

At (0, -1)

$$0^2 + (-1)^2 + 2g(0) + 2f(-1) + c = 0$$

$$0g - 2f + c = -1 \quad \dots\dots\dots (3)$$

(1) + (2)

$$8f + 2c = -10 \quad \dots\dots\dots (4)$$

(3) x 2

$$-4f + 2c = -2 \quad \dots\dots\dots (5)$$

(4) - (5)

$$12f = -8$$

$$f = \frac{-8}{12}$$

$$f = -\frac{2}{3} \quad \dots\dots\dots (6)$$

(6) into (5)

$$-4\left(-\frac{2}{3}\right) + 2c = -2$$

$$\frac{8}{3} + 2c = -2$$

$$2c = -2 - \frac{8}{3}$$

$$2c = -\frac{14}{3}$$

$$c = -\frac{7}{3} \quad \dots\dots\dots (7)$$

Substitute (6) & (7) into (2)

$$-2g + 4\left(-\frac{2}{3}\right) - \frac{7}{3} = -5$$

$$-2g - \frac{8}{3} - \frac{7}{3} = -5$$

$$-2g - \frac{15}{3} = -5$$

$$-2g = -5 + \frac{15}{3}$$

$$-2g = 0$$

$$g = 0$$

General Equation of circle:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$x^2 + y^2 + 2(0)x + 2\left(-\frac{2}{3}\right)y - \frac{7}{3} = 0$$

$$x^2 + y^2 - \frac{4}{3}y - \frac{7}{3} = 0$$

Center of circle,

$$C = (-g, -f)$$

$$= \left(-0, -\left(-\frac{2}{3}\right)\right)$$

$$= \left(0, \frac{2}{3}\right)$$

2. Show that $\int_1^e x \ln x \, dx = \frac{1}{4}(1 + e^2)$.

SOLUTION

$$\int x \ln x \, dx =$$

$$u = \ln x$$

$$dv = x \, dx$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\int dv = \int x \, dx$$

$$du = \frac{1}{x} dx$$

$$v = \frac{x^2}{2}$$

$$\int u \, dv = uv - \int v \, du$$

$$\int x \ln x \, dx = (\ln x) \left(\frac{x^2}{2} \right) - \int \left(\frac{x^2}{2} \right) \left(\frac{1}{x} dx \right)$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4}$$

$$\int_1^e x \ln x \, dx = \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} \right]_1^e$$

$$= \left[\frac{e^2}{2} (\ln e) - \frac{e^2}{4} \right] - \left[\frac{1^2}{2} (\ln 1) - \frac{1^2}{4} \right]$$

$$= \left[\frac{e^2}{2} - \frac{e^2}{4} \right] - \left[0 - \frac{1}{4} \right]$$

$$= \left[\frac{2e^2}{4} - \frac{e^2}{4} \right] + \frac{1}{4}$$

$$= \frac{e^2 + 1}{4}$$

$$= \frac{1}{4}(1 + e^2)$$

3. Find y in terms of x given that $x \frac{dy}{dx} = (1 - 2x^2)y$ where $x > 0$ and $y = 1$ when $x = 1$.

SOLUTION

$$x \frac{dy}{dx} = (1 - 2x^2)y$$

$$\frac{dy}{y} = \frac{(1 - 2x^2)}{x} dx$$

$$\frac{dy}{y} = \left(\frac{1}{x} - \frac{2x^2}{x} \right) dx$$

$$\frac{1}{y} dy = \left(\frac{1}{x} - 2x \right) dx$$

$$\int \frac{1}{y} dy = \int \left(\frac{1}{x} - 2x \right) dx$$

$$\ln y = \ln x - x^2 + C$$

When $x = 1, y = 1$

$$\ln 1 = \ln 1 - 1^2 + C$$

$$0 = 0 - 1 + C$$

$$C = 1$$

Particular Solution

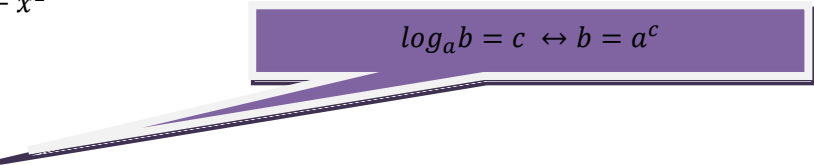
$$\ln y = \ln x - x^2 + 1$$

$$\ln y - \ln x = 1 - x^2$$

$$\ln \left(\frac{y}{x} \right) = 1 - x^2$$

$$\frac{y}{x} = e^{1-x^2}$$

$$y = xe^{1-x^2}$$



$$\log_a b = c \leftrightarrow b = a^c$$

4. Find the general solution of the differential equation $\frac{dy}{dx} + y \cot x = 2 \sin x$.

SOLUTION

$$\frac{dy}{dx} + y \cot x = 2 \sin x$$

Compare to

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$P(x) = \cot x = \frac{\cos x}{\sin x}$$

$$Q(x) = 2 \sin x$$

Integrating factor,

$$V(x) = e^{\int P(x)dx}$$

$$= e^{\int \left(\frac{\cos x}{\sin x}\right) dx}$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)|$$

$$= e^{\ln(\sin x)}$$

$$= \sin x$$

$$V(x)y = \int V(x)Q(x)dx$$

$$(\sin x)y = \int (\sin x)(2 \sin x)dx$$

$$y \sin x = 2 \int \sin^2 x dx$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$y \sin x = 2 \int \frac{1 - \cos 2x}{2} dx$$

$$y \sin x = \int 1 - \cos 2x dx$$

$$y \sin x = x - \frac{\sin 2x}{2} + C$$

5. Express $\frac{1-4x}{3+x-2x^2}$ in partial fractions and hence, find the exact value of $\int_0^1 \frac{1-4x}{3+x-2x^2} dx$.

SOLUTION

$$\frac{1-4x}{3+x-2x^2} = \frac{1-4x}{(3-2x)(1+x)}$$

$$\frac{1-4x}{3+x-2x^2} = \frac{A}{(3-2x)} + \frac{B}{(1+x)}$$

$$\frac{1-4x}{3+x-2x^2} = \frac{A(1+x) + B(3-2x)}{(3-2x)(1+x)}$$

$$1-4x = A(1+x) + B(3-2x)$$

When $x = -1$

$$1-4(-1) = A(1-1) + B[3-2(-1)]$$

$$5 = 5B$$

$$B = 1$$

When $x = \frac{3}{2}$

$$1-4\left(\frac{3}{2}\right) = A\left(1+\frac{3}{2}\right) + B\left[3-2\left(\frac{3}{2}\right)\right]$$

$$-5 = \frac{5}{2}A$$

$$A = -2$$

$$\therefore \frac{1-4x}{3+x-2x^2} = \frac{-2}{(3-2x)} + \frac{1}{(1+x)}$$

$$\int \frac{1-4x}{3+x-2x^2} dx = \int \frac{-2}{(3-2x)} + \frac{1}{(1+x)} dx$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)|$$

$$\int \frac{1-4x}{3+x-2x^2} dx = \ln(3-2x) + \ln(1+x)$$
$$= \ln(3-2x)(1+x)$$

$$\ln a + \ln b = \ln(ab)$$

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

$$\int_0^1 \frac{1-4x}{3+x-2x^2} dx = [\ln(3-2x)(1+x)]_0^1$$
$$= [\ln(3-2(1))(1+1)] - [\ln(3-2(0))(1+0)]$$
$$= [\ln 2] - [\ln 3]$$
$$= \ln\left(\frac{2}{3}\right)$$

6. (a) Given $f_1(x) = 2x$ and $f_2(x) = -\ln x$.
- Without using curve sketching, show that $y = f_1(x)$ and $y = f_2(x)$ intersect on the interval of $[0.1, 1]$.
 - Use Newton-Raphson's method to estimate the intersection point of $y = f_1(x)$ and $y = f_2(x)$, with the initial value $x_1 = 1$. Iterate until $|f(x_n)| < 0.005$. Give your answer correct to three decimal places.
- b) By using the trapezoidal rule, find the approximate value for $\int_0^1 x\sqrt{x+1} dx$ when $n = 4$, correct to four decimal places.

SOLUTION**(ai)**

$$f_1(x) = 2x, \quad f_2(x) = -\ln x$$

$$\text{Let } f(x) = f_1(x) - f_2(x)$$

$$f(x) = 2x - (-\ln x)$$

$$f(x) = 2x + \ln x$$

$$f(0.1) = 2(0.1) + \ln(0.1) = -2.013 < 0$$

$$f(1) = 2(1) + \ln 1 = 2 > 0$$

Since $f(0.1) < 0$ and $f(1) > 0$, by using **intermediate value theorem**, there are at least one root in the interval $[0.1, 1]$.

Intermediate Value Theorem

http://mathinsight.org/intermediate_value_theorem_location_roots_refresher

(aii) Newton-Raphson's method

$$f(x) = 2x + \ln x$$

$$f'(x) = 2 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{2x_n + \ln x_n}{2 + \frac{1}{x_n}}$$

$$x_1 = 1$$

$$x_2 = 1 - \frac{2(1) + \ln(1)}{2 + \frac{1}{(1)}} = 0.3333$$

$$x_3 = 0.3333 - \frac{2(0.3333) + \ln(0.3333)}{2 + \frac{1}{(0.3333)}} = 0.4197$$

$$x_4 = 0.4197 - \frac{2(0.4197) + \ln(0.4197)}{2 + \frac{1}{(0.4197)}} = 0.4263$$

$$x_5 = 0.4263 - \frac{2(0.4263) + \ln(0.4263)}{2 + \frac{1}{(0.4263)}} = 0.4263$$

$$\therefore x = 0.426$$

When $x = 0.426$

$$f_1(x) = 2x$$

$$f_1(0.426) = 2(0.426) = 0.852$$

$\therefore (0.426, 0.852)$ is the intersection point of $f_1(x)$ and $f_2(x)$

(b) Trapezoidal Rule

$$\int_0^1 x\sqrt{x+1} dx \text{ when } n = 4$$

$$h = \frac{1-0}{4} = 0.25$$

Trapezoidal Rule

$$\int_a^b f(x) dx \approx \frac{h}{2} [y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})]$$

$$h = \frac{b-a}{n}$$

y_n	x_n	$f(x) = x\sqrt{x+1}$	
y_0	0.00	0.00000	
y_1	0.25		0.27951
y_2	0.50		0.61237
y_3	0.75		0.99216
y_4	1.00	1.41421	
Total		$(y_0 + y_n) = 1.41421$	$(y_1 + y_2 + \dots + y_{n-1}) = 1.88404$

By trapezoidal rule:

$$\int_0^1 x\sqrt{x+1} dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

$$= \frac{0.25}{2} [(1.41421) + 2(1.88404)]$$

$$= 0.64779$$

$$= 0.6478$$

7. (a) If $\mathbf{p} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{q} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, show that

$$|\mathbf{p} \times \mathbf{q}|^2 = |\mathbf{p}|^2 |\mathbf{q}|^2 - (\mathbf{p} \cdot \mathbf{q})^2$$

- (b) Given a triangle ABC with $\overrightarrow{AB} = 2\mathbf{a}$ and $\overrightarrow{AC} = 3\mathbf{b}$. Use the result in part (a), show that the area of the triangle is $3\sqrt{|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2}$. Hence, deduce the area of the triangle if $\mathbf{a} = \mathbf{p}$ and $\mathbf{b} = \mathbf{q}$

SOLUTION

$$(a) \mathbf{p} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \mathbf{q} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\begin{aligned} \mathbf{p} \times \mathbf{q} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix} \\ &= [(1) - (4)]\mathbf{i} - [(-3) - (4)]\mathbf{j} + [(6) - (-2)]\mathbf{k} \\ &= -3\mathbf{i} - (-7)\mathbf{j} + (8)\mathbf{k} \\ &= -3\mathbf{i} + 7\mathbf{j} + 8\mathbf{k} \end{aligned}$$

$$\begin{aligned} |\mathbf{p} \times \mathbf{q}| &= \sqrt{(-3)^2 + (7)^2 + (8)^2} \\ &= \sqrt{122} \end{aligned}$$

$$|\mathbf{p} \times \mathbf{q}|^2 = 122$$

$$\begin{aligned} |\mathbf{p}| &= \sqrt{(3)^2 + (-1)^2 + (2)^2} \\ &= \sqrt{14} \end{aligned}$$

$$|\mathbf{p}|^2 = 14$$

$$\begin{aligned} |\mathbf{q}| &= \sqrt{(2)^2 + (2)^2 + (-1)^2} \\ &= \sqrt{9} \end{aligned}$$

$$|\mathbf{q}|^2 = 9$$

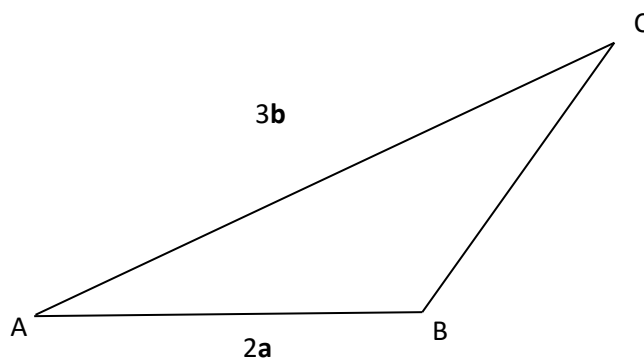
$$\begin{aligned} \mathbf{p} \cdot \mathbf{q} &= (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\ &= (3)(2) + (-1)(2) + (2)(-1) \end{aligned}$$

$$\mathbf{p} \cdot \mathbf{q} = 2$$

$$\begin{aligned} |\mathbf{p}|^2 |\mathbf{q}|^2 - (\mathbf{p} \cdot \mathbf{q})^2 &= (14)(9) - (2)^2 \\ &= 122 \end{aligned}$$

$$\therefore |\mathbf{p} \times \mathbf{q}|^2 = |\mathbf{p}|^2 |\mathbf{q}|^2 - (\mathbf{p} \cdot \mathbf{q})^2$$

(b)



$$\text{Area, } A = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$A = \frac{1}{2} |2\mathbf{a} \times 3\mathbf{b}|$$

$$= 3|\mathbf{a} \times \mathbf{b}|$$

$$= 3|\mathbf{a} \times \mathbf{b}|$$

$$= 3\sqrt{|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2}$$

$$(m\mathbf{a} \times \mathbf{b}) = m(\mathbf{a} \times \mathbf{b})$$

$$2\mathbf{a} \times 3\mathbf{b} = (2)(\mathbf{a} \times 3\mathbf{b})$$

$$= (2 \times 3)(\mathbf{a} \times \mathbf{b})$$

$$= (6)(\mathbf{a} \times \mathbf{b})$$

When $\mathbf{a} = \mathbf{p}$ and $\mathbf{b} = \mathbf{q}$

$$\text{Area} = 3\sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2}$$

$$= 3\sqrt{|\mathbf{p}|^2|\mathbf{q}|^2 - (\mathbf{p} \cdot \mathbf{q})^2}$$

$$= 3\sqrt{122}$$

8. Given a line $l: x = 2 - t, y = -3 + 4t, z = -5 - 3t$, and two planes $\pi_1: 2x - y + 7z = 53$ and $\pi_2: 3x + y + z = 1$. Find
- The point of intersection between the line l and the plane π_1 .
 - The acute angle between the line l and the plane π_1 .
 - The acute angle between planes π_1 and π_2 .

SOLUTION

$$l: x = 2 - t, \quad y = -3 + 4t, \quad z = -5 - 3t,$$

$$\pi_1: 2x - y + 7z = 53$$

$$\pi_2: 3x + y + z = 1$$

(a)

$$x = 2 - t, \quad y = -3 + 4t, \quad z = -5 - 3t \quad \dots\dots\dots (1)$$

$$2x - y + 7z = 53 \quad \dots\dots\dots (2)$$

Substitute (1) into (2)

$$2(2 - t) - (-3 + 4t) + 7(-5 - 3t) = 53$$

$$4 - 2t + 3 - 4t - 35 - 21t = 53$$

$$-28 - 27t = 53$$

$$27t = -81$$

$$t = -3$$

When $t = -3$

$$x = 2 - t = 2 + 3 = 5$$

$$y = -3 + 4t = -3 - 12 = -15$$

$$z = -5 - 3t = -5 + 9 = 4$$

\therefore the intersection point is $(5, -15, 4)$

(b)

$$x = 2 - t, \quad y = -3 + 4t, \quad z = -5 - 3t \quad \dots\dots\dots (1)$$

$$2x - y + 7z = 53 \quad \dots\dots\dots (2)$$

Direction vector of the line:

$$\mathbf{v} = -\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$$

Normal vector of the plane:

$$\mathbf{n} = 2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$$

Angle Between The Line l And The Plane π_1

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{v}| \cdot |\mathbf{n}|}$$

$$\begin{aligned} \mathbf{v} \cdot \mathbf{n} &= (-\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} + 7\mathbf{k}) \\ &= (-1)(2) + (4)(-1) + (-3)(7) \\ &= -27 \end{aligned}$$

$$\begin{aligned} |\mathbf{v}| &= \sqrt{(-1)^2 + (4)^2 + (-3)^2} \\ &= \sqrt{26} \end{aligned}$$

$$\begin{aligned} |\mathbf{n}| &= \sqrt{(2)^2 + (-1)^2 + (7)^2} \\ &= \sqrt{54} \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{v}| \cdot |\mathbf{n}|} \\ &= \frac{-27}{\sqrt{26} \cdot \sqrt{54}} \\ &= -0.7206 \end{aligned}$$

$$\begin{aligned}\theta &= \cos^{-1} \frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{v}| \cdot |\mathbf{n}|} \\ &= \cos^{-1}(-0.7206) \\ &= 136.10^\circ\end{aligned}$$

Acute Angle Between The Line l And The Plane π_1

$$\begin{aligned}\alpha &= 136.10^\circ - 90^\circ \\ &= 46.10^\circ\end{aligned}$$

(c)

$$\pi_1: 2x - y + 7z = 53 \qquad \pi_2: 3x + y + z = 1$$

Angle Between The Plane π_1 And The Plane π_2

$$\theta = \cos^{-1} \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \cdot |\mathbf{n}_2|}$$

$$\mathbf{n}_1 = 2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$$

$$\mathbf{n}_2 = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\begin{aligned}\mathbf{n}_1 \cdot \mathbf{n}_2 &= (2\mathbf{i} - \mathbf{j} + 7\mathbf{k})(3\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= (2)(3) + (-1)(1) + (7)(1) \\ &= 12\end{aligned}$$

$$\begin{aligned}|\mathbf{n}_1| &= \sqrt{(2)^2 + (-1)^2 + (7)^2} \\ &= \sqrt{54}\end{aligned}$$

$$\begin{aligned}|\mathbf{n}_2| &= \sqrt{(3)^2 + (1)^2 + (1)^2} \\ &= \sqrt{11}\end{aligned}$$

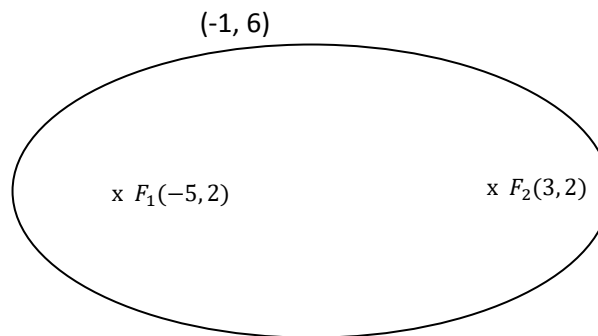
Angle Between The Plane π_1 And The Plane π_2

$$\begin{aligned}\theta &= \cos^{-1} \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \cdot |\mathbf{n}_2|} \\ &= \cos^{-1} \frac{12}{\sqrt{54} \cdot \sqrt{11}} \\ &= \cos^{-1}(0.4924) \\ &= 60.50^\circ\end{aligned}$$

9. (a) Find the equation in standard form of an ellipse which passes through the point $(-1, 6)$ and having foci at $(-5, 2)$ and $(3, 2)$.
- (b) From the result obtained in part (a), sketch the graph of the ellipse.

SOLUTION

(a)

*Equation of the ellipse:*

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\text{where } a^2 - b^2 = c^2$$

$$\text{Centre, } C(h, k) = \left(\frac{-5 + 3}{2}, \frac{2 + 2}{2} \right)$$

$$= \left(\frac{-2}{2}, \frac{4}{2} \right)$$

$$= (-1, 2)$$

Distance between two foci:

$$2c = 3 - (-5)$$

$$= 8$$

$$c = 4$$

$$a^2 - b^2 = c^2$$

$$a^2 - b^2 = 4^2$$

$$a^2 - b^2 = 16 \quad \dots\dots\dots (1)$$

Equation of the ellipse:

$$\frac{(x + 1)^2}{a^2} + \frac{(y - 2)^2}{b^2} = 1$$

At $(-1, 6)$

$$\frac{(-1 + 1)^2}{a^2} + \frac{(6 - 2)^2}{b^2} = 1$$

$$0 + \frac{16}{b^2} = 1$$

$$b^2 = 16 \quad \dots\dots\dots (2)$$

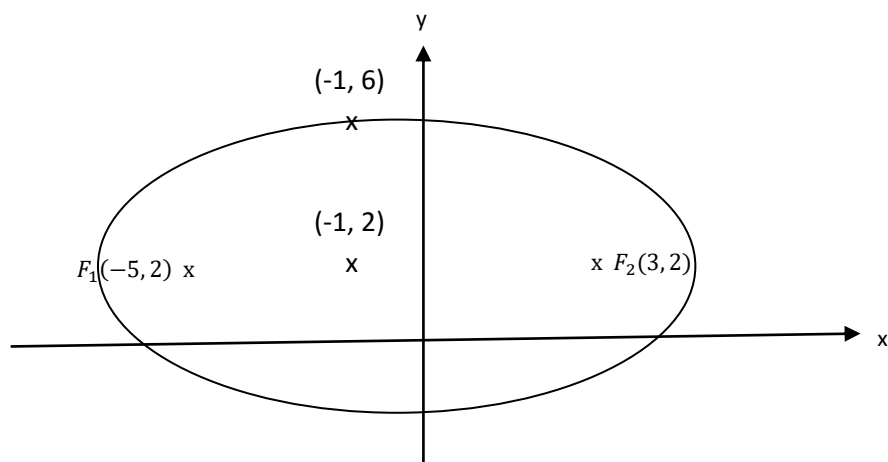
Substitute (2) into (1)

$$a^2 - 16 = 16$$

$$a^2 = 32$$

$$\therefore \frac{(x + 1)^2}{32} + \frac{(y - 2)^2}{16} = 1$$

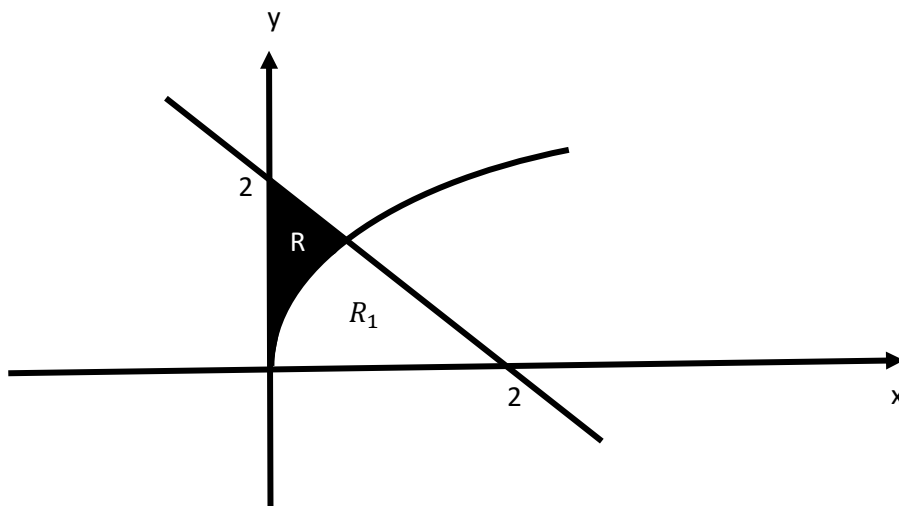
(b)



10. (a) Sketch and shade the region R bounded by the curve $y = \sqrt{x}$, line $y = 2 - x$ and y -axis. Hence, find the area of the region R .
- (b) If R_1 is a region bounded by the curve $y = \sqrt{x}$, line $y = 2 - x$ and x -axis, deduce the ratio of $R: R_1$.
- (c) Find the volume of the solid generated when the region R is rotated through 360° about the x -axis.

SOLUTION**(a)**

$$y = \sqrt{x}, \quad y = 2 - x$$



$$y = \sqrt{x} \quad \dots\dots\dots (1)$$

$$y = 2 - x \quad \dots\dots\dots (2)$$

Substitute (1) into (2)

$$\sqrt{x} = 2 - x$$

$$x = (2 - x)^2$$

$$x = 4 + x^2 - 4x$$

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

$$x = 1 \text{ or } x = 4$$

$$\therefore x = 1$$

$$\begin{aligned} \text{Area, } R &= \int_0^1 (2 - x) - \sqrt{x} \, dx \\ &= \int_0^1 2 - x - x^{\frac{1}{2}} \, dx \\ &= \left[2x - \frac{x^2}{2} - \frac{2x^{\frac{3}{2}}}{3} \right]_0^1 \\ &= \left[2(1) - \frac{1^2}{2} - \frac{2(1)^{\frac{3}{2}}}{3} \right] - \left[2(0) - \frac{0^2}{2} - \frac{2(0)^{\frac{3}{2}}}{3} \right] \\ &= \left[2 - \frac{1}{2} - \frac{2}{3} \right] - 0 \\ &= \frac{5}{6} \end{aligned}$$

(b)

$$\begin{aligned} \text{Area } R_1 &= \frac{1}{2}(2)(2) - \frac{5}{6} \\ &= \frac{7}{6} \end{aligned}$$

Ratio of $R:R_1$

$$\begin{aligned} \frac{R}{R_1} &= \frac{\frac{5}{6}}{\frac{7}{6}} \\ &= \frac{5}{6} \times \frac{6}{7} \\ &= \frac{5}{7} \end{aligned}$$

$$\therefore R:R_1 = 5:7$$

(c)

Volume of the solid generated when the region R is rotated through 360° about the x -axis.

$$\begin{aligned} \text{Volume, } V &= \pi \int_0^1 (2-x)^2 - (\sqrt{x})^2 dx \\ &= \pi \int_0^1 (4+x^2-4x) - (x) dx \\ &= \pi \int_0^1 (x^2-5x+4) dx \\ &= \pi \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x \right]_0^1 \\ &= \pi \left[\left(\frac{(1)^3}{3} - \frac{5(1)^2}{2} + 4(1) \right) - \left(\frac{(0)^3}{3} - \frac{5(0)^2}{2} + 4(0) \right) \right] \\ &= \pi \left[\left(\frac{1}{3} - \frac{5}{2} + 4 \right) - 0 \right] \\ &= \frac{11}{6} \pi \end{aligned}$$