

QS 025

Mid-Semester Examination

Semester II

Session 2012/2013

- 1) Find
 - a) $\int_0^1 \frac{3+e^x}{e^x} dx$.
 - b) $\int \sin^2 x \cos x dx$
- 2) Using integration by parts, solve
$$\int 2x \sin 2x dx$$
- 3) Find the particular solution of the differential equation
$$(x+2) \frac{dy}{dx} = y+2, \quad y(0) = 1$$
- 4) Show that the equation $2x + e^x - 2 = 0$ has a root between $x = 0$ and $x = 1$. Using the Newton-Raphson method and taking $x_0 = 0.7$, find the root correct to four decimal places.
- 5) (a) Using partial fractions, find $\int \frac{3x^2-2x+2}{(x-1)(x^2+2)} dx$.
(b) Given functions $f(x) = x + 7$ and $g(x) = x^2 - 5$, find the area of the region bounded by $f(x)$ and $g(x)$.
- 6) A circle passes through the points $A(-2, 4)$, $B(7, 7)$ and the centre lies on the line $y - x = 1$. Find the radius and standard equation of the circle.

END OF QUESTION PAPER

1) Find

a) $\int_0^1 \frac{3+e^x}{e^x} dx.$

b) $\int \sin^2 x \cos x dx$

SOLUTION

(a)

$$\int \frac{3+e^x}{e^x} dx = \int \frac{3}{e^x} + \frac{e^x}{e^x} dx$$

$$= \int (3e^{-x} + 1) dx$$

$$= -3e^{-x} + x$$

$$= \frac{-3}{e^x} + x$$

$$\int_0^1 \frac{3+e^x}{e^x} dx = \left[\frac{-3}{e^x} + x \right]_0^1$$

$$= \left[\frac{-3}{e^1} + 1 \right] - \left[\frac{-3}{e^0} + 0 \right]$$

$$= \left[\frac{-3}{e} + 1 \right] - \left[\frac{-3}{1} + 0 \right]$$

$$= \frac{-3}{e} + 1 + 3$$

$$= 4 - \frac{3}{e}$$

(b)

$$\int \sin^2 x \cos x dx$$

$$u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$dx = \frac{1}{\cos x} du$$

$$\int \sin^2 x \cos x dx = \int u^2 \cos x \left(\frac{1}{\cos x} du \right)$$

$$= \int u^2 du$$

$$= \frac{u^3}{3} + c$$

$$= \frac{\sin^3 x}{3} + c$$

2) Using integration by parts, solve

$$\int 2x \sin 2x \, dx$$

SOLUTION

$$\int 2x \sin 2x \, dx$$

$$u = 2x$$

$$dv = \sin 2x \, dx$$

$$\frac{du}{dx} = 2$$

$$\int dv = \int \sin 2x \, dx$$

$$du = 2dx$$

$$v = \frac{-\cos 2x}{2}$$

$$\int u \, dv = uv - \int v \, du$$

$$\int 2x \sin 2x \, dx = (2x) \left(\frac{-\cos 2x}{2} \right) - \int \left(\frac{-\cos 2x}{2} \right) (2dx)$$

$$= -x \cos 2x + \int \cos 2x \, dx$$

$$= -x \cos 2x + \frac{\sin 2x}{2} + c$$

3) Find the particular solution of the differential equation

$$(x + 2) \frac{dy}{dx} = y + 2, \quad y(0) = 1$$

SOLUTION

$$(x + 2) \frac{dy}{dx} = y + 2$$

$$(x + 2)dy = (y + 2)dx$$

$$\frac{1}{(y + 2)} dy = \frac{1}{(x + 2)} dx$$

$$\int \frac{1}{(y + 2)} dy = \int \frac{1}{(x + 2)} dx$$

$$\ln|y + 2| = \ln|x + 2| + c$$

Given that $y(0) = 1 \rightarrow$ when $x = 0, y = 1$

$$\ln|1 + 2| = \ln|0 + 2| + c$$

$$\ln|3| = \ln|2| + c$$

$$c = \ln|3| - \ln|2|$$

$$c = \ln\left(\frac{3}{2}\right)$$

$$\log m - \log n = \log\left(\frac{m}{n}\right)$$

$$\ln|y + 2| = \ln|x + 2| + c$$

$$\ln|y + 2| = \ln|x + 2| + \ln\left(\frac{3}{2}\right)$$

$$\ln|y + 2| = \ln\frac{3|x + 2|}{2}$$

$$\log m + \log n = \log(mn)$$

$$y + 2 = \frac{3}{2}(x + 2)$$

$$y = \frac{3}{2}(x + 2) - 2$$

$$y = \frac{3x}{2} + 3 - 2$$

$$y = \frac{3x}{2} + 1$$

- 4) Show that the equation $2x + e^x - 2 = 0$ has a root between $x = 0$ and $x = 1$. Using the Newton-Raphson method and taking $x_0 = 0.7$, find the root correct to four decimal places.

SOLUTION

$$2x + e^x - 2 = 0$$

$$f(x) = 2x + e^x - 2$$

$$f(0) = 2(0) + e^{(0)} - 2 = -1 < 0$$

$$f(1) = 2(1) + e^{(1)} - 2 = e > 0$$

Since $f(0) < 0$ and $f(1) > 0$. $\therefore f(x)$ has a root between $x = 0$ and $x = 1$.

$$f(x) = 2x + e^x - 2$$

$$f'(x) = 2 + e^x$$

$$x_0 = 0.7$$

$$x_1 = 0.7 - \frac{2(0.7) + e^{0.7} - 2}{2 + e^{0.7}} = 0.34777$$

$$x_2 = 0.34777 - \frac{2(0.34777) + e^{0.34777} - 2}{2 + e^{0.34777}} = 0.31514$$

$$x_3 = 0.31514 - \frac{2(0.31514) + e^{0.31514} - 2}{2 + e^{0.31514}} = 0.31492$$

$$x_4 = 0.31492 - \frac{2(0.31492) + e^{0.31492} - 2}{2 + e^{0.31492}} = 0.31492$$

\therefore The root is 0.3149

5) (a) Using partial fractions, find $\int \frac{3x^2-2x+2}{(x-1)(x^2+2)} dx$.

(b) Given functions $f(x) = x + 7$ and $g(x) = x^2 - 5$, find the area of the region bounded by $f(x)$ and $g(x)$.

SOLUTION

(a)

$$\frac{3x^2 - 2x + 2}{(x - 1)(x^2 + 2)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 2}$$

$$\frac{3x^2 - 2x + 2}{(x - 1)(x^2 + 2)} = \frac{A(x^2 + 2) + (Bx + C)(x - 1)}{(x - 1)(x^2 + 2)}$$

$$3x^2 - 2x + 2 = A(x^2 + 2) + (Bx + C)(x - 1)$$

When $x = 1$

$$3(1)^2 - 2(1) + 2 = A[(1)^2 + 2] + [B(1) + C][(1) - 1]$$

$$3 = 3A$$

$$A = 1$$

When $x = 0$, $A = 1$

$$3(0)^2 - 2(0) + 2 = 1[(0)^2 + 2] + [B(0) + C][(0) - 1]$$

$$2 = 2 - C$$

$$C = 0$$

When $x = -1$, $A = 1$, $C = 0$

$$3(-1)^2 - 2(-1) + 2 = 1[(-1)^2 + 2] + [B(-1) + 0][(-1) - 1]$$

$$7 = 3 + [-B][-2]$$

$$4 = 2B$$

$$B = 2$$

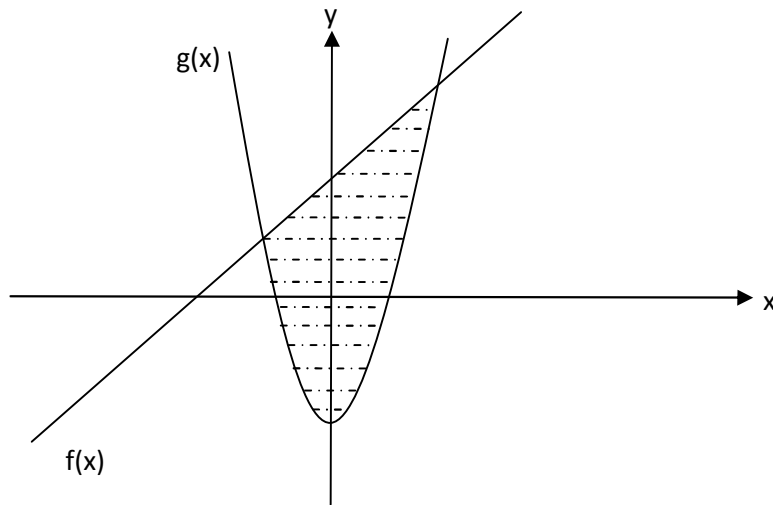
$$\frac{3x^2 - 2x + 2}{(x - 1)(x^2 + 2)} = \frac{1}{x - 1} + \frac{2x}{x^2 + 2}$$

$$\int \frac{3x^2 - 2x + 2}{(x - 1)(x^2 + 2)} dx = \int \frac{1}{x - 1} + \frac{2x}{x^2 + 2} dx$$

$$= \ln(x - 1) + \ln(x^2 + 2) + C$$

(b)

$$f(x) = x + 7 \text{ and } g(x) = x^2 - 5$$



$$f(x) = x + 7 \quad \dots\dots\dots (1)$$

$$g(x) = x^2 - 5 \quad \dots\dots\dots (2)$$

To find intersection points, let $f(x) = g(x)$.

$$x + 7 = x^2 - 5$$

$$x^2 - x - 12 = 0$$

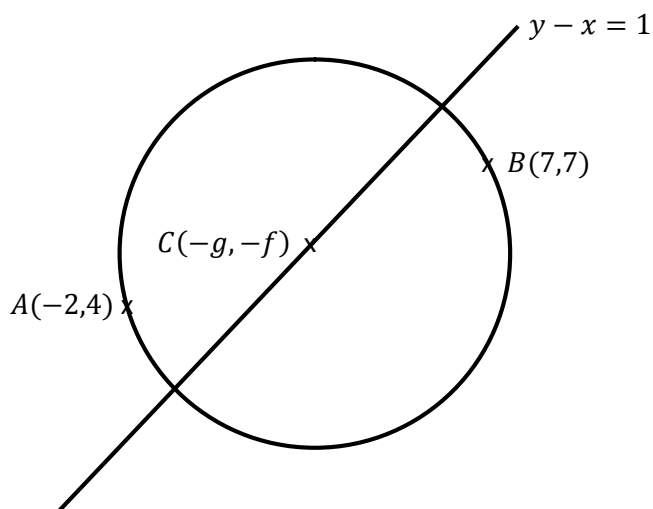
$$(x - 4)(x + 3) = 0$$

$$x = 4 \quad \text{or} \quad x = -3$$

$$\begin{aligned} \text{Area} &= \int_{-3}^4 (x + 7) - (x^2 - 5) \, dx \\ &= \int_{-3}^4 -x^2 + x + 12 \, dx \\ &= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 12x \right]_{-3}^4 \\ &= \left[-\frac{4^3}{3} + \frac{4^2}{2} + 12(4) \right] - \left[-\frac{(-3)^3}{3} + \frac{(-3)^2}{2} + 12(-3) \right] \\ &= \frac{343}{6} \text{ unit}^2 \end{aligned}$$

- 6) A circle passes through the points $A(-2, 4)$, $B(7, 7)$ and the centre lies on the line $y - x = 1$. Find the radius and standard equation of the circle.

SOLUTION



General Equation:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

At the point $A(-2, 4) \rightarrow x = -2; \quad y = 4$

$$(-2)^2 + (4)^2 + 2g(-2) + 2f(4) + c = 0$$

$$4 + 16 - 4g + 8f + c = 0$$

$$4g - 8f - c = 20 \quad \dots\dots\dots (1)$$

At the point $B(7, 7) \rightarrow x = 7; \quad y = 7$

$$(7)^2 + (7)^2 + 2g(7) + 2f(7) + c = 0$$

$$49 + 49 + 14g + 14f + c = 0$$

$$14g + 14f + c = -98 \quad \dots\dots\dots (2)$$

Center $C(-g, -f)$ lies on the line $y - x = 1 \rightarrow x = -g, \quad y = -f$

$$-f - (-g) = 1$$

$$g - f = 1 \quad \dots\dots\dots (3)$$

(2) - (1)

$$10g + 22f = -118 \quad \dots\dots\dots (4)$$

(3) x 10

$$10g - 10f = 10 \quad \dots\dots\dots (5)$$

$$(4) - (5)$$

$$32f = -128$$

$$f = -4$$

$$10g - 10(-4) = 10$$

$$10g = -30$$

$$g = -3$$

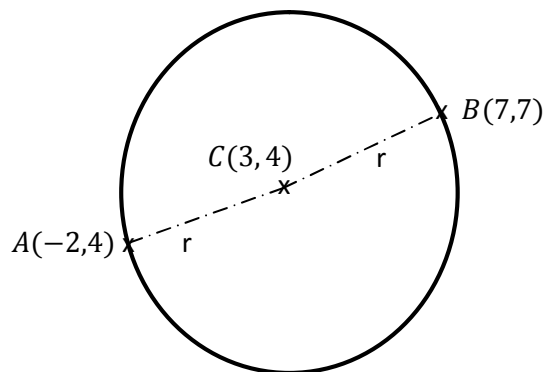
$$4g - 8f - c = 20$$

$$4(-3) - 8(-4) - c = 20$$

$$20 - c = 20$$

$$c = 0$$

\therefore Center $(3, 4)$



$A(-2, 4)$, Center $(3, 4)$

$$\begin{aligned} \text{radius} &= \sqrt{(3 + 2)^2 + (4 - 4)^2} \\ &= 5 \end{aligned}$$

Standard equation of circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 3)^2 + (y - 4)^2 = 5^2$$

$$(x - 3)^2 + (y - 4)^2 = 25$$