

QS 025

Mid-Semester Examination

Semester II

Session 2014/2015

1. Find the equation of the parabola with vertex $(-1,3)$ and focus $(-1,4)$. Sketch the graph of the parabola by labelling the vertex, focus and directrix.
2. Show that the equation $16x^2 + 4y^2 - 64x - 40y + 100 = 0$ represents an ellipse. Find the centre, foci and vertices of the ellipse.
3. Find the general solution of the linear differential equation $\frac{dy}{dx} + \frac{y}{x} = x$ by expressing y in terms of x .
4. Use the trapezoidal rule to find an approximate value for $\int_0^2 \frac{1}{1+x^2} dx$ with four sub-intervals.
5. By using partial fractions, show that
$$\frac{x}{x^2 - 1} = \frac{1}{2(x + 1)} + \frac{1}{2(x - 1)}$$
Hence, evaluate $\int_2^5 \frac{x}{x^2 - 1} dx$.
6. (a) By using substitution $u = \sqrt{x - 1}$, find $\int \frac{x}{\sqrt{x-1}} dx$.
(b) Find $\int xe^{4x} dx$.

1. Find the equation of the parabola with vertex $(-1,3)$ and focus $(-1,4)$. Sketch the graph of the parabola by labelling the vertex, focus and directrix.

SOLUTION

$$v(-1,3), f(-1,4)$$

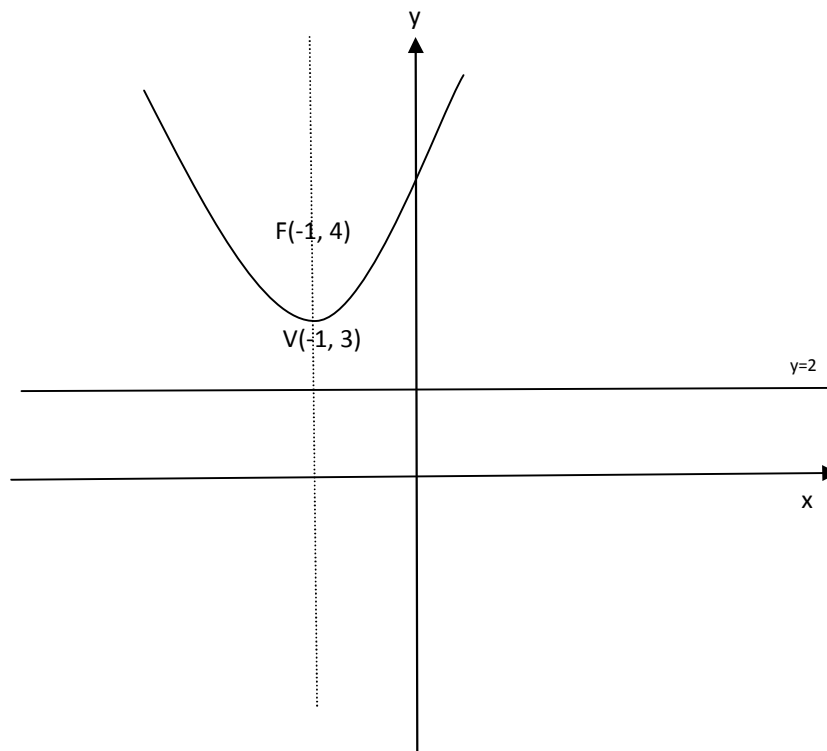
$$h = -1, k = 3$$

$$p = 4 - 3 = 1$$

$$(x - h)^2 = 4p(y - k)$$

$$(x + 1)^2 = 4(1)(y - 3)$$

$$(x + 1)^2 = 4(y - 3)$$



2. Show that the equation $16x^2 + 4y^2 - 64x - 40y + 100 = 0$ represents an ellipse.
Find the centre, foci and vertices of the ellipse.

SOLUTION

$$16x^2 + 4y^2 - 64x - 40y + 100 = 0$$

$$16x^2 - 64x + 4y^2 - 40y = -100$$

$$16(x^2 - 4x) + 4(y^2 - 10y) = -100$$

$$16 \left[x^2 - 4x + \left(\frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2 \right] + 4 \left[y^2 - 10y + \left(\frac{-10}{2}\right)^2 - \left(\frac{-10}{2}\right)^2 \right] = -100$$

$$16[x^2 - 4x + (-2)^2 - (-2)^2] + 4[y^2 - 10y + (-5)^2 - (-5)^2] = -100$$

$$16[(x - 2)^2 - 4] + 4[(y - 5)^2 - 25] = -100$$

$$16(x - 2)^2 - 64 + 4(y - 5)^2 - 100 = -100$$

$$16(x - 2)^2 + 4(y - 5)^2 = -100 + 64 + 100$$

$$16(x - 2)^2 + 4(y - 5)^2 = 64$$

$$\frac{16(x - 2)^2}{64} + \frac{4(y - 5)^2}{64} = \frac{64}{64}$$

$$\frac{(x - 2)^2}{4} + \frac{(y - 5)^2}{16} = 1$$

Compare

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

$$h = 2 \quad k = 5$$

$$a^2 = 4 \quad b^2 = 16$$

$$a = 2 \quad b = 4$$

$$c^2 = b^2 - a^2$$

$$c^2 = 16 - 4$$

$$c = \sqrt{12}$$

$$\text{Centre: } (h, k) = (2, 5)$$

$$V_1(h, k+b) = V_1(2, 5+4) = V_1(2, 9)$$

$$V_2(h, k-b) = V_2(2, 5-4) = V_2(2, 1)$$

$$F_1(h, k+c) = F_1(2, 5+\sqrt{12})$$

$$F_2(h, k-c) = F_2(2, 5-\sqrt{12})$$

3. Find the general solution of the linear differential equation $\frac{dy}{dx} + \frac{y}{x} = x$ by expressing y in terms of x .

SOLUTION

$$\frac{dy}{dx} + \frac{y}{x} = x$$

$$\frac{dy}{dx} + \frac{1}{x}(y) = x$$

$$P(x) = \frac{1}{x}, \quad Q(x) = x$$

$$V(x) = e^{\int P(x)dx}$$

$$V(x) = e^{\int \frac{1}{x} dx}$$

$$V(x) = e^{\ln(x)}$$

$$V(x) = x$$

$$V(x)y = \int V(x)Q(x)dx$$

$$xy = \int x(x)dx$$

$$xy = \int x^2 dx$$

$$xy = \frac{x^3}{3} + c$$

$$y = \frac{x^3}{3x} + \frac{c}{x}$$

$$y = \frac{x^2}{3} + \frac{c}{x}$$

$$\frac{dy}{dx} + \frac{1}{x}(y) = x$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

4. Use the trapezoidal rule to find an approximate value for $\int_0^2 \frac{1}{1+x^2} dx$ with four sub-intervals.

SOLUTION

$$\int_0^2 \frac{1}{1+x^2} dx$$

$$a = 0, b = 2, n = 4, f(x) = \frac{1}{1+x^2}$$

$$h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$$

X	$f(x) = \frac{1}{1+x^2}$	
0.0	1.0	
0.5		0.8000
1.0		0.5000
1.5		0.3077
2.0	0.2	
Total	1.2	1.6077

$$\int_0^2 \frac{1}{1+x^2} dx = \frac{h}{2} [(y_0 + y_1) + 2(y_1 + y_2 + y_3)]$$

$$\int_0^2 \frac{1}{1+x^2} dx = \frac{0.5}{2} [(1.2) + 2(1.6077)]$$

$$\int_0^2 \frac{1}{1+x^2} dx = 1.1039$$

5. By using partial fractions, show that

$$\frac{x}{x^2 - 1} = \frac{1}{2(x + 1)} + \frac{1}{2(x - 1)}$$

Hence, evaluate $\int_2^5 \frac{x}{x^2 - 1} dx$.

SOLUTION

$$\frac{x}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

$$\frac{x}{(x + 1)(x - 1)} = \frac{A(x - 1) + B(x + 1)}{(x + 1)(x - 1)}$$

$$x = A(x - 1) + B(x + 1)$$

When $x = 1$:

$$1 = A(1 - 1) + B(1 + 1)$$

$$2B = 1$$

$$B = \frac{1}{2}$$

When $x = -1$:

$$-1 = A(-1 - 1) + B(-1 + 1)$$

$$-2A = -1$$

$$A = \frac{1}{2}$$

$$\therefore \frac{x}{x^2 - 1} = \frac{1}{2(x + 1)} + \frac{1}{2(x - 1)}$$

$$\int_2^5 \frac{x}{x^2 - 1} dx = \int_2^5 \frac{1}{2(x + 1)} + \frac{1}{2(x - 1)} dx$$

$$\int_2^5 \frac{x}{x^2 - 1} dx = \frac{1}{2} \int_2^5 \frac{1}{(x + 1)} + \frac{1}{(x - 1)} dx$$

$$= \frac{1}{2} [\ln(x + 1) + \ln(x - 1)]_2^5$$

$$\begin{aligned} &= \frac{1}{2} [\ln(x+1)(x-1)]_2^5 \\ &= \frac{1}{2} [\ln(5+1)(5-1) - \ln(2+1)(2-1)] \\ &= \frac{1}{2} [\ln(6)(4) - \ln(3)(1)] \\ &= \frac{1}{2} [\ln 24 - \ln 3] \\ &= \frac{1}{2} \left[\ln \frac{24}{3} \right] \\ &= \frac{1}{2} [\ln 8] \\ &= 1.040 \end{aligned}$$

6. (a) By using substitution $u = \sqrt{x-1}$, find $\int \frac{x}{\sqrt{x-1}} dx$.
 (b) Find $\int xe^{4x} dx$.

SOLUTION

6(a)

$$\int \frac{x}{\sqrt{x-1}} dx$$

$$u = \sqrt{x-1} \rightarrow u^2 = x-1 \rightarrow x = u^2 + 1$$

$$u = (x-1)^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x-1}}$$

$$\frac{du}{dx} = \frac{1}{2u}$$

$$dx = 2udu$$

$$\int \frac{x}{\sqrt{x-1}} dx = \int \frac{u^2 + 1}{u} (2udu)$$

$$= 2 \int u^2 + 1 du$$

$$= 2 \left[\frac{u^3}{3} + u \right] + c$$

$$= \frac{2u^3}{3} + 2u + c$$

$$= \frac{2(\sqrt{x-1})^3}{3} + 2\sqrt{x-1} + c$$

6(b)

$$\int xe^{4x} dx$$

$$u = x \quad dv = e^{4x} dx$$

$$du = dx \quad v = \int e^{4x} dx$$

$$v = \frac{e^{4x}}{4}$$

$$\int x e^{4x} dx = \frac{x e^{4x}}{4} - \int \frac{e^{4x}}{4} dx$$

$$\int x e^{4x} dx = \frac{x e^{4x}}{4} - \frac{e^{4x}}{16} + c$$